Maximum and Minimum Values 3

Introduction

In Maximum and Minimum Values 2 you will have seen that $\frac{dy}{dx} = 0$ at a turning point on the graph of $y = f(x)$, and that an examination of the gradient pattern close to the turning point may be used to discriminate between a maximum and a minimum point.

However, a more accurate test for the nature of a turning point uses the second derivative.

Second Derivative

The result of differentiating once is called the **first derivative** and is written as $\frac{dy}{dx}$.

The result of differentiating the expression obtained for $\frac{dy}{dx}$ is called the **second derivative** and is written $\frac{d^2y}{dx^2}$.

Determining the Nature of the Turning Points

We can use the second derivative to decide if the turning point that we have located is a maximum or a minimum:

When the second derivative, evaluated at the turning point, is less than zero we always have a maximum.

When the second derivative, evaluated at the turning point, is greater than zero we always have a minimum.

$$\frac{d^2y}{dx^2} < 0 \quad \text{indicates a maximum}$$

$$\frac{d^2y}{dx^2} > 0 \quad \text{indicates a minimum}$$

Example
The demand equation of a good is $P + Q = 30$

Determine the value of Q which will maximise the total revenue (TR) given by $TR = PQ$.

Solution

The function we wish to maximise is $TR = PQ$.

We are trying to find the value of Q which maximises this function.

So, first we express TR wholly in terms of Q

Given that $P + Q = 30$ it follows that $P = 30 - Q$ and so,

$TR = (30 - Q)Q = 30Q - Q^2$

In order to find the maximum value of TR we must first determine $\dfrac{d(TR)}{dQ}$

Differentiation gives $\dfrac{d(TR)}{dQ} = 30 - 2Q$

At a turning point this derivative must be zero, so $30 - 2Q = 0$

It follows that $Q = 15$ at this turning point.

Since the second derivative $\dfrac{d}{dQ}(30 - 2Q) = -2$ is less than 0 it follows that when $Q = 15$ we have a maximum for TR.

Exercises

1. The demand equation of a good is $P + 2Q = 20$. Determine the value of Q necessary for maximum total revenue, TR, given by $TR = PQ$. Verify that this value of Q produces a maximum value for TR.

2. A profit (π) function is given by $\pi = -1.5Q^2 + 24Q - 7$. Determine the value of Q that maximises this profit. Verify that this value of Q produces a maximum value for π.

3. The average running costs (£C) for a removal company is related to the number of vans (v) in use by the expression $C = \dfrac{162}{v} + 4.5v + 125$.

   Determine the number of vans which will ensure minimum running costs.

   Verify that this value of v produces a minimum value for C.

4. A long distance haulage firm has determined that the cost (£C) of running its lorries on a particular journey is related to the average speed (x mph) of the lorry by the expression $C = \dfrac{9075}{x} + 3x$

   Determine the optimum speed which will ensure running costs are kept to a minimum.

   Verify that this value of x produces a minimum value for C.

Answers

(1) $Q = 5$, $\dfrac{d^2(TR)}{dQ^2} = -4$, max (2) $Q = 8$, $\dfrac{d^2\pi}{dQ^2} = -3$, max (3) $v = 6$, $\dfrac{d^2C}{dv^2} = 1.5$, min

(4) $x = 55$, $\dfrac{d^2C}{dx^2} = 6$, min

