The Algebra of Polynomials

By definition a polynomial (in the indeterminate x) is just an expression

$$p(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

where the terms $a_n,\ a_{n-1}\ \ldots\ a_1,\ a_0$, called coefficients, are all real constants. For example, in the case of the polynomial

$$p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 6$$

the coefficients are $a_3 = 7,\ a_2 = -12,\ a_1 = -8$ and $a_0 = 6$.

Generally, we will think of the polynomial $p(x)$ above as a function:

$$p : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto p(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

Thus, in the case of $p(x) = 7\,x^3 - 12\,x^2 - 8\,x + 6$, we can, for example, evaluate at $x = 0$ to find

$$p(0) = 7\,(0)^3 - 12\,(0)^2 - 8\,(0) + 6 \;=\; 6.$$

While evaluating at $x = -2$ gives

$$p(-2) = 7\,(-2)^3 - 12\,(-2)^2 - 8\,(-2) + 6 \;=\; -82.$$

The **highest power** of x that appears in $p(x)$ is called the **degree** of $p(x)$. Thus,

$$\text{degree}\left(a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0\right) = n,$$

while

$$\text{degree}\left(7\,x^3 - 12\,x^2 - 8\,x + 6\right) = 3,$$

Addition of Polynomials: We add polynomials (in the indeterminate x) by adding the coefficients of corresponding powers of x. For example:

$$\begin{aligned}
&(3\,x^5 - 7\,x^3 + 9\,x^2 - x + 6) + (9\,x^5 - 3\,x^4 - 8\,x^2 + 5\,x - 10) \\
=\ &(3+9)\,x^5 \ +\ (0-3)\,x^4 \ +\ (-7+0)\,x^3 \ +\ (9-8)\,x^2 \ +\ (-1+5)\,x \ +\ (6-10) \\
=\ &12\,x^5 \ -\ 3\,x^4 \ -\ 7\,x^3 \ +\ x^2 \ +\ 4\,x \ -\ 4\,.
\end{aligned}$$

Multiplication of Polynomials: When multiplying polynomials (which we think of as real-valued functions) the usual rules for multiplication of real numbers apply. Thus, we have:

(i) $(a\,x^n)\,(b\,x^m) = (a\,b)\,x^{n+m}$.

(ii) For any polynomials $p(x)$, $q(x)$ and $h(x)$;

$$[\,p(x) + q(x)\,]\,h(x) = p(x)\,h(x) + q(x)\,h(x)$$

(iii) For any polynomials $p(x)$ and $q(x)$; $\quad p(x)\,q(x) = q(x)\,p(x)$

We illustrate multiplication of polynomials by the following example:

$$\begin{aligned}
&(\,3\,x^5 - 7\,x^2 + 9\,x + 6\,)\,(\,9\,x^3 - 8\,x^2 + 5\,x - 10\,) \\
= \;& 3\,x^5\,(\,9\,x^3 - 8\,x^2 + 5\,x - 10\,) - 7\,x^2\,(\,9\,x^3 - 8\,x^2 + 5\,x - 10\,) + \\
& \qquad 9\,x\,(\,9\,x^3 - 8\,x^2 + 5\,x - 10\,) + 6\,(\,9\,x^3 - 8\,x^2 + 5\,x - 10\,) \\
= \;& (\,27\,x^8 - 24\,x^7 + 15\,x^6 - 30\,x^5) + (\,-\,63\,x^5 + 56\,x^4 - 35\,x^3 + 70\,x^2\,) + \\
& \qquad (\,81\,x^4 - 72\,x^3 + 45\,x^2 - 90\,x\,) + (\,54\,x^3 - 48\,x^2 + 30\,x - 60\,) \\
= \;& 27\,x^8 - 24\,x^7 + 15\,x^6 - 93\,x^5 + 137\,x^4 - 53\,x^3 + 67\,x^2 - 60\,x - 60.
\end{aligned}$$

Note: It should be clear from the foregoing that, for any polynomials $p(x)$ and $q(x)$, we have

$$\text{degree}(\,p(x)\,q(x)\,) = \text{degree}(\,p(x)\,) + \text{degree}(\,q(x)\,)$$

Long Division of polynomials: As you learned in school, you can divide a polynomial $p(x)$ by the polynomial $q(x)$ through the process of long division. Schematically the process of long division has the form:

$$\begin{array}{r|l}
 & \;s(x) \\ \hline
q(x) & \;p(x) \qquad\qquad\qquad \\
 & \qquad\qquad \ddots \quad \underline{\qquad\quad} \\
 & \qquad\qquad\qquad\quad r(x)
\end{array}$$

The process stops when:

$$\deg r(x) < \deg q(x),$$

and the polynomial $r(x)$ is called **the remainder**.

For example, let’s divide $p(x) = (7x^5 - x^2 + 4x + 6)$ by $q(x) = (x^3 - 2x + 5)$.

$$
\begin{array}{r|l}
 & 7x^2 + \mathbf{14} \\ \hline
(x^3 - 2x + 5) & 7x^5 - x^2 + 4x + 6 \\
 & 7x^5 - 14x^3 + 35x^2 \\ \hline
 & \qquad 14x^3 - 36x^2 + 4x + 6 \\
 & \qquad \mathbf{14x^3 - 28x + 70} \\ \hline
 & \qquad\qquad -36x^2 + 32x - 64
\end{array}
$$

Here

$$\deg(-36x^2 + 32x - 64) = 2 < 3 = \deg(x^3 - 2x + 5)$$

so the process stops. What we have shown is that:

$$(7x^5 - x^2 + 4x + 6) = (x^3 - 2x + 5)(7x^2 + 14) + (-36x^2 + 32x - 64)$$

In general, what the long division:

$$
\begin{array}{r|l}
 & s(x) \\ \hline
q(x) & p(x) \\
 & \quad \ddots \\ \cline{2-2}
 & \qquad\qquad r(x)
\end{array}
$$

shows is that

$$\boxed{\begin{aligned} p(x) &= q(x)\,s(x) + r(x)\,, \\ \text{where:}\quad \deg r(x) &< \deg q(x). \end{aligned}}\;. \tag{1}$$

Note: The reason for the term division is that we can rewrite the above in the form:

$$\boxed{\begin{aligned} \frac{p(x)}{q(x)} &= s(x) + \frac{r(x)}{q(x)}\,, \\ \text{where:}\quad \deg r(x) &< \deg q(x). \end{aligned}}\;.$$

However, this latter form will not be as useful to us as will be the former.

Important Special Case: A very important special case of (1) above arises when

$$q(x) = (x - \xi)\,, \quad \text{for some constant } \xi.$$

In this case, $\deg q(x) = 1$ and, therefore, $\deg r(x) = 0$. Hence, $r(x) = c$ for some constant $= c$. Writing (1) in this context, we obtain that:

$$p(x) \;=\; (x - \xi)\, s(x) + c\,.$$

To determine the constant c, simply put $x = \xi$ into this latter equation to find that:

$$p(\xi) \;=\; (\xi - \xi)\, s(\xi) + c\,.$$

That is:

$$p(\xi) \;=\; \mathbf{0} + c\,.$$

so that $c = p(\xi)$ and we have proved:

The Remainder Theorem: For any polynomial $p(x)$ and for any constant ξ we have that:

$$p(x) \;=\; (x - \xi)\, s(x) + p(\xi)\,,$$

for some polynomial $s(x)$.

An immediate consequence of the Remainder Theorem is that for any polynomial $p(x)$:

$$\boxed{p(\xi) \;=\; 0} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c} p(x) \;=\; (x - \xi)\, s(x) \\ \text{for some polynomial } s(x) \end{array}}\,.$$

Or put in words

$$\boxed{\xi \text{ is a root of } p(x)} \quad \Longleftrightarrow \quad \boxed{(x - \xi) \text{ is a factor of } p(x)}\,.$$