# The Comma of Pythagoras

In school you learned how to sing the major scale:

Do, Re, Mi, Fa, So, La, Ti, Do.

Typically, this would be sung in C major, where the starting note (i.e. Do) is C, usually the middle C on the piano. The notes then, given in corresponding order, are:

C, D, E, F, G, A, B, C.

Since we will use the terms below, we remark that:

The final C is said to be an octave (i.e. 8 notes) above the initial C.

Here the initial and final C's are counted. Similarly,

The G is said to be a fifth (i.e. 5 notes) above the initial C.

For the very able singers, this pattern repeats itself and some can sing in higher and higher octaves.

C, D, E, F, G, A, B, C, D, E, F, G, A, B, C.

Others, mostly big beefy men, can go in the other direction and sing in lower octaves.

There are certain physical reasons why notes which are an octave apart are always in harmony with each other and are, in many respects, regarded as being the same note. For similar physical reasons, notes where one is a fifth above the other also harmonize, though not in the same way as in the case of octaves.

Consider the fiddle, if you were to play the scale

C, D, E, F, G, A, B, C,

on a fiddle you would notice that the finger placements from C to D and from D to E are further apart than are those from E to F. In fact, the pattern is as follows:

$$\text{C} \quad \text{tone} \quad \text{D} \quad \text{tone} \quad \text{E} \left\{ \begin{array}{c} \text{semi-} \\ \text{tone} \end{array} \right\} \text{F} \quad \text{tone} \quad \text{G} \quad \text{tone} \quad \text{A} \quad \text{tone} \quad \text{B} \left\{ \begin{array}{c} \text{semi-} \\ \text{tone} \end{array} \right\} \text{C.}$$

In fact, extra notes are added to split the tones in half. Thus, the distance between any two consecutive notes is now be a semi-tone. This is the basic musical interval in classical western music. The names and notation given to the extra notes depends on the context, but in this context they are denoted as follows:

C, $C^{\#}$, D, $D^{\#}$, E, F , $F^{\#}$, G, $G^{\#}$, A, $A^{\#}$, B, C.

The notes $C^{\#}$, $D^{\#}$ , $F^{\#}$, $G^{\#}$, $A^{\#}$ are called C-sharp, D-sharp, F-sharp, etc. In other contexts, they are referred to as flats. For example, $A^{\#}$ would be denoted by $B^{\flat}$ and called $B - flat$. The other sharps/flats are treated similarly.

**Observation:** Starting at the first C you will count 12 semi-tones to reach the C an octave higher. Similarly, you will count 7 semi-tones to reach the G which is a fifth above the starting C. In summary:

$$\begin{aligned} \text{One octave} &= \text{12 semi-tones} \\ \text{One fifth} &= \text{7 semi-tones} \end{aligned}$$

**Question:** Suppose you sing, starting at C and progressing ever higher in intervals of fifths, what notes do you sing? In particular, do you ever hit a C, which is some number of octaves, above your starting C?

**Answer:** Imagine that we play all of the notes (a semi-tone apart) on a single fiddle string. The notes, as we increase the pitch, are:

C, $C^{\#}$, D, $D^{\#}$, E, F , $F^{\#}$, G, $G^{\#}$, A, $A^{\#}$, B, C , $C^{\#}$, D, $D^{\#}$, E, F , ...

We list the notes (starting at C) which we encounter as we go up in fifths. That is, the notes which are successively 7 semi-tones above one another. These are:

C, G, D, A, E, B, $F^{\#}$, $C^{\#}$, $G^{\#}$ , ...

If we are going to reach some C which is n octaves above our starting C by counting up a total of m fifths then we must have

$$12\,n = 7\,m.$$

Clearly, since $n$ and $m$ are integers and since 7 and 12 have no common factors, we must have that

$$n \text{ is a multiple of 7} \quad \text{and} \quad m \text{ is a multiple of 12}.$$

The smallest possible such $n$ and $m$ are

$$n = 7 \quad \text{and} \quad m = 12.$$

Thus, if we start at C and play 12 notes each a fifth above the preceding note then we will play for the first time again a C, and this C is, 7 octaves above our starting C. All of this seems to be grand! But is it?

### The Problem: Where are the notes located?

Imagine that you are bowing a fiddle string without any finger on the string. Now, place a finger on the string half way between the nut and the bridge and continue to bow the half nearest the bridge. You will find that there is perfect harmony between this note and the note that you were playing on the full length string. By definition, (according to Pythagoras) the note played on the half string is an octave above that played on the full string. Similarly, if you place a finger a distance L/3 up from the nut (where L is the length of the string) and if you bow the remaining piece nearest the bridge (which has length 2L/3) you will again find that there is harmony between this note and the note that you were playing on the full length string. According to Pythagoras, the note played on two thirds of the string is a fifth above that played on the full string. To find the second fifth we would have to bow the piece of string nearest the bridge which has length two thirds of two thirds of L that is of length $(2/3)^2$L. In general we have:

| Note played | length of string bowed |
|---|---|
| base note | L |
| first fifth | $(2/3)$L |
| second fifth | $(2/3)^2$L |
| third fifth | $(2/3)^3$L |
| ... | ... |
| ... | ... |
| twelfth fifth | $(2/3)^{12}$L |

The same argument for octaves gives:

| Note played | length of string bowed |
|---|---|
| base note | L |
| first octave | $(1/2)$L |
| second octave | $(1/2)^2$L |
| third octave | $(1/2)^3$L |
| ... | ... |
| ... | ... |
| seventh octave | $(1/2)^7$L |

Accordingly, if the 12th fifth is to be the same as the 7th octave then we must have:

$$(2/3)^{12}\mathrm{L} = (1/2)^7\mathrm{L}$$

or

$$2^{19} = 3^{12},$$

which is impossible.

Notice that if the string has length 1 (1 meter, say) then the 12th fifth is played by placing your finger a distance $(1 - (2/3)^{12}) = 0.992293\ldots$ from the nut. On the other hand, the 7th octave is played by placing your finger a distance $(1 - (1/2)^7) = 0.992188\ldots$ from the nut. Since

$$0.992293\ldots > 0.992188\ldots$$

it follows that the 12th fifth is sharp compared to the 7th octave. It is this difference, (or the ratio $\{(2/3)^{12}\}/\{(1/2)^7\}$) which is called **The Comma of Pythagoras**. For more, see (for instance):

http://www.ericweisstein.com/encyclopedias/music/CommaofPythagoras.html