The Decimal Representation of Real Numbers

Motivation : **The decimal notation used to represent numbers is just a short-hand notation. For example, the symbol**

$$847\,.\,359862$$

is the decimal notation for the number

$$8 \times 10^2 + 4 \times 10^1 + 7 \times 10^0 + \frac{3}{10^1} + \frac{5}{10^2} + \frac{9}{10^3} + \frac{8}{10^4} + \frac{6}{10^5} + \frac{2}{10^6}\,.$$

In general, if $a_0, a_1, \ldots a_n$ **are digits (i.e. numbers between 0 and 9, inclusive) and likewise** $b_1, b_2, \ldots b_m$**, then the symbol**

$$a_n a_{n-1} \,\ldots a_0\,.\,b_1 b_2\,\ldots b_m\,,$$

is the decimal representation of the number

$$a_n \times 10^n + a_{n-1} \times 10^{n-1} + \ldots + a_0 \times 10^0 + \frac{b_1}{10^1} + \frac{b_2}{10^2} + \ldots + \frac{b_m}{10^m}\,.$$

Note: The fact that we're working in base 10 is of no importance. We could just as well work in any other base, for example in base 2.

As you have learned in school, to obtain decimal representations of fractions we proceed as in the following examples:

Example 1:

$$\frac{7}{5} = 1 + \frac{2}{5} \qquad \textbf{The remainder is 2}$$

$$= 1 + \left[\frac{2}{5}\right] \cdot \frac{10}{10}$$

$$= 1 + \left[\frac{20}{5}\right] \cdot \frac{1}{10}$$

$$= 1 + \left[4 + \frac{0}{5}\right] \cdot \frac{1}{10} \qquad \textbf{The remainder is 0}$$

$$= 1 + \frac{4}{10}.$$

That is, in decimal notation $\frac{7}{5} = 1.4$

Example 2:

$$
\begin{aligned}
\frac{5}{16} &= \frac{5}{16} \cdot \frac{\mathbf{10}}{\mathbf{10}} \\
&= \frac{\mathbf{50}}{16} \cdot \frac{1}{\mathbf{10}} \\
&= \left[3 + \frac{2}{16}\right] \cdot \frac{1}{\mathbf{10}} && \textbf{The remainder is 2} \\
&= \frac{3}{\mathbf{10}} + \left[\frac{2}{16}\right] \cdot \frac{1}{\mathbf{10}} \\
&= \frac{3}{\mathbf{10}} + \left[\frac{\mathbf{20}}{16}\right] \cdot \frac{1}{\mathbf{10^2}} \\
&= \frac{3}{\mathbf{10}} + \left[1 + \frac{4}{16}\right] \cdot \frac{1}{\mathbf{10^2}} && \textbf{The remainder is 4} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \left[\frac{4}{16}\right] \cdot \frac{1}{\mathbf{10^2}} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \left[\frac{\mathbf{40}}{16}\right] \cdot \frac{1}{\mathbf{10^3}} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \left[2 + \frac{8}{16}\right] \cdot \frac{1}{\mathbf{10^3}} && \textbf{The remainder is 8} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \frac{2}{\mathbf{10^3}} + \left[\frac{8}{16}\right] \cdot \frac{1}{\mathbf{10^3}} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \frac{2}{\mathbf{10^3}} + \left[\frac{\mathbf{80}}{16}\right] \cdot \frac{1}{\mathbf{10^4}} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \frac{2}{\mathbf{10^3}} + \left[5 + \frac{0}{16}\right] \cdot \frac{1}{\mathbf{10^4}} && \textbf{The remainder is 0} \\
&= \frac{3}{\mathbf{10}} + \frac{1}{\mathbf{10^2}} + \frac{2}{\mathbf{10^3}} + \frac{5}{\mathbf{10^4}}
\end{aligned}
$$

Thus, $\frac{5}{16} = 0\,.\,3125$ **in decimal notation.**

Example 3:

$$\frac{3}{7} = 0 + \frac{3}{7}$$ **The remainder is 3**

$$= \frac{30}{7} \cdot \frac{1}{10}$$

$$= \left[4 + \frac{2}{7}\right] \cdot \frac{1}{10}$$ **The remainder is 2**

$$= \frac{4}{10} + \left[\frac{20}{7}\right] \cdot \frac{1}{10^2}$$

$$= \frac{4}{10} + \left[2 + \frac{6}{7}\right] \cdot \frac{1}{10^2}$$ **The remainder is 6**

$$= \frac{4}{10} + \frac{2}{10^2} + \left[\frac{60}{7}\right] \cdot \frac{1}{10^3}$$

$$= \frac{4}{10} + \frac{2}{10^2} + \left[8 + \frac{4}{7}\right] \cdot \frac{1}{10^3}$$ **The remainder is 4**

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \left[\frac{40}{7}\right] \cdot \frac{1}{10^4}$$

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \left[5 + \frac{5}{7}\right] \cdot \frac{1}{10^4}$$ **The remainder is 5**

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \left[\frac{50}{7}\right] \cdot \frac{1}{10^5}$$

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \left[7 + \frac{1}{7}\right] \cdot \frac{1}{10^5}$$ **The remainder is 1**

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \left[\frac{10}{7}\right] \cdot \frac{1}{10^6}$$

$$= \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \left[1 + \frac{3}{7}\right] \cdot \frac{1}{10^6}$$ **The remainder 3 appears again**

Thus, we have found that

$$\frac{3}{7} = \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{1}{10^6} + \left[\frac{3}{7}\right] \cdot \frac{1}{10^6} \qquad (1)$$

If we continue as above, then the same pattern will repeat itself identically. That is, we will get both the same sequence of digits (in blue) which give us the decimal places and the same sequence of remainders (in magenta). In particular,

> **the remainder zero never occurs so the process never stops.**

Indeed, on replacing the bracketed term $\left[\frac{3}{7}\right]$ **in (1) above by the full expression for** $\frac{3}{7}$ **itself, [i.e. by the full expression in (1)], we find that**

$$\begin{aligned} \frac{3}{7} \quad = \quad & \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{1}{10^6} + \\ & \left[\frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{1}{10^6} + \left[\frac{3}{7}\right] \cdot \frac{1}{10^6}\right] \cdot \frac{1}{10^6} \end{aligned}$$

That is,

$$\begin{aligned} \frac{3}{7} \quad = \quad & \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{1}{10^6} + \\ & \frac{4}{10^7} + \frac{2}{10^8} + \frac{8}{10^9} + \frac{5}{10^{10}} + \frac{7}{10^{11}} + \frac{1}{10^{12}} + \left[\frac{3}{7}\right] \cdot \frac{1}{10^{12}} \end{aligned}$$

and on using equation (1) again we have

$$\frac{3}{7} = \frac{4}{10} + \frac{2}{10^2} + \frac{8}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{1}{10^6} +$$
$$\frac{4}{10^7} + \frac{2}{10^8} + \frac{8}{10^9} + \frac{5}{10^{10}} + \frac{7}{10^{11}} + \frac{1}{10^{12}} +$$
$$\frac{4}{10^{13}} + \frac{2}{10^{14}} + \frac{8}{10^{15}} + \frac{5}{10^{16}} + \frac{7}{10^{17}} + \frac{1}{10^{18}} + \left[\frac{3}{7}\right] \cdot \frac{1}{10^{18}}$$

Accordingly, in decimal notation we have

$$\frac{3}{7} = 0\,.\underbrace{428571}\underbrace{428571}\underbrace{428571} + \left[\frac{3}{7}\right] \cdot \frac{1}{10^{18}}$$

We can continue in this way to find that

$$\frac{3}{7} = 0\,.\underbrace{428571}\underbrace{428571}\underbrace{428571}\underbrace{428571}\ \ldots$$

In particular, even to represent rational numbers by decimals we must allow decimals which consist of infinite strings.

From what we have shown above, it should be clear that the infinite string decimals which represent rationals are of a very special form.