1 The Derivative

2.5 Definition

The *derivative* of the function f is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if the limit exists).

The domain of f' is the set of points in the domain of f for which the limit exists.

If f' exists at a particular x, we say that f is differentiable at x.

If f' exists at every point of the domain of f, we call f differentiable.

The process of calculating the derivative f' of a particular function f is called differentiation.

We saw in the last section that

If $f(x) = x^2$, then $f'(x) = 2x$.

If $f(x) = \frac{1}{x}$, then $f'(x) = -\frac{1}{x^2}$.

Notation
Let $y = f(x)$. The following different notations are often used for the derivative $f'(x)$:

$$f'(x) = \begin{cases} y' & \text{"}y - prime\text{"}; \\ \frac{dy}{dx} & \text{"}d - y - d - x\text{"}; \\ \frac{df}{dx} & \text{"}d - f - d - x\text{"}; \\ \frac{d}{dx}(f(x)) & \text{"}d - d - x \text{ of } f(x)\text{"}. \end{cases}$$

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2.6 Examples

(i) Let $g(x) = 3x^2$. Then

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h} \\ &= \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} \\ &= \lim_{h \to 0} (6x + h) = 6x. \end{aligned}$$

Note if $f(x) = x^2$, then $g(x) = 3f(x)$ and

$$g'(x) = 6x = 3f'(x).$$

(ii) Let $h(x) = x$ then

$$\begin{aligned} h'(x) &= \lim_{h \to 0} \frac{x + h - x}{h} \\ &= \lim_{h \to 0} 1 = 1. \end{aligned}$$

Note the graph of h is a straight line with slope 1.

(iii) Let $k(x) = x^2 + x$. Then

$$\begin{aligned} k'(x) &= \lim_{h \to 0} \frac{(x+h)^2 + x + h - x^2 - x}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h} \\ &= \lim_{h \to 0} (2x + 1 + h) = 2x + 1. \end{aligned}$$

Note if $f(x) = x^2$, $h(x) = x$ then $k(x) = f(x) + h(x)$ and

$$k'(x) = 2x + 1 = f'(x) + h'(x).$$

2.7 Rules for Derivatives

(i) If $f(x)$ is a constant function, say $f(x) = c$, then $f'(x) = 0$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 = 0. \end{aligned}$$

Power rule for positive integers

(ii) If $f(x) = x^n$, where n is a natural number (positive whole number), then

$$f'(x) = nx^{n-1}.$$

We already know that this is true for $n = 1, 2$.

If $n = 3$, then $f(x) = x^3$ and

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2. \end{aligned}$$

We will prove it by induction later on for any n. We can apply this rule to get the following:

$$\begin{aligned} f(x) = x^7 \quad &\Longrightarrow \quad f'(x) = 7x^6; \\ g(x) = x^5 \quad &\Longrightarrow \quad g'(x) = 5x^4; \\ h(x) = x^{128} \quad &\Longrightarrow \quad h'(x) = 128x^{127}. \end{aligned}$$

Constant Multiple Rule

(iii) If c is any constant and u is a differentiable function of x, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

$$\begin{aligned} (cu)'(x) = \quad & \lim_{h\to 0} \frac{(cu)(x+h) - (cu)(x)}{h} \\ &= \lim_{h\to 0} \frac{c(u(x+h)) - c(u(x))}{h} \\ &= c \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= c(u'(x)). \end{aligned}$$

Sum Rule

(iv) If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at very point where u and v are both differentiable and

$$
\begin{aligned}
(u+v)'(x) &= \lim_{h\to 0} \frac{(u+v)(x+h)-(u+v)(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\
&= u'(x)+v'(x).
\end{aligned}
$$

Difference Rule

(v) Combining (iii) and (iv) we get the difference rule.

If u and v are differentiable functions of x, then their difference $u-v$ is differentiable at very point where u and v are both differentiable and

$$(u-v)'(x) = (u+(-v))'(x) = u'(x) + (-v)'(x) = u'(x) - v'(x).$$

2.8 Derivative of a Polynomial

Putting these rules together, and as we saw in the examples: If

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$ and $n > 0$, then

$$f'(x) = na_n x^{n-1} + \cdots + a_1.$$

2.9 Example

Find the horizontal tangents to the curve $y = x^4 - 2x^2 + 2$.

2.10 Second and Higher-Order Derivatives

If $y = x^4 - x^2 + 6x - 1,$ then

$$y' = 4x^3 - 2x + 6.$$

If $u = 4x^3 - 2x + 6,$ then

$$u' = 12x^2 - 2.$$

If $v = 12x^2 - 2$, then

$$v' = 24x.$$

Now $u = y'$ and $v = u' = (y')'$. These are called higher-order derivatives.

The derivative $y' = \dfrac{dy}{dx}$ is the first (first-order) derivative of y with respect to x.

The derivative may itself be a differentiable function of x; if so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the second (second-order) derivative of y with respect to x.

If y'' is differentiable, its derivative

$$y^{(3)} = y''' = \frac{dy''}{dx} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

is called the third (third-order) derivative of y with respect to x.

This can continue with

$$y^{(n)} = \frac{d}{dx}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right) = \frac{d^ny}{dx^n}$$

is called the nth (nth-order) derivative of y with respect to x.

More Notation

$$y' \qquad \text{“}y - \text{prime”};$$

$$y'' \qquad \text{“}y - \text{double prime”};$$

$$\tfrac{d^2y}{dx^2} \qquad \text{“}d \text{ squared } y\ dx \text{ squared”};$$

$$y''' \qquad \text{“}y - \text{triple prime”};$$

$$y^{(n)} \qquad \text{“}y - \text{super } n\text{”};$$

$$\tfrac{d^ny}{dx^n} \qquad \text{“}d \text{ to the } n \text{ of } y \text{ by } dx \text{ to the } n\text{”}.$$