

0.1 Derivatives of Products, Quotients and Negative Powers

2.11 Products

The derivative of the sum of two functions is the sum of their derivatives, the derivative of the product is *not* the product of their derivatives. For example

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x$$

whereas

$$\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1.$$

The derivative of a product of two functions is in fact the sum of two products.

Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

or

$$(uv)' = uv' + vu'.$$

2.12 Example

(i) Let $f(x) = (5x^2 + 1)(2x - 2)$.

Set $u(x) = 5x^2 + 1$ and $v(x) = 2x - 2$.

Use the product rule $(uv)' = uv' + vu'$.

$$u'(x) = 10x, v'(x) = 2$$

Therefore by the product rule

$$f'(x) = (5x^2 + 1)2 + (2x - 2)10x = 30x^2 - 20x + 2.$$

(ii) Let $f(x) = (2x^3 + x)(x^2 - x)$.

Set $u(x) = 2x^3 + x$ and $v(x) = x^2 - x$

Use the product rule $(uv)' = uv' + vu'$.

$$u'(x) = 6x^2 + 1, v'(x) = 2x - 1.$$

By the product rule

$$\begin{aligned} f'(x) &= (2x^3 + x)(2x - 1) + (x^2 - x)(6x^2 + 1) \\ &= 10x^4 - 8x^3 + 3x^2 - 2x. \end{aligned}$$

Proof of Product rule

$$\begin{aligned} (uv)' &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h)(v(x+h) - v(x))}{h} + \lim_{h \rightarrow 0} \frac{v(x)(u(x+h) - u(x))}{h} \\ &= \lim_{h \rightarrow 0} u(x+h) \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} + v(x) \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \\ &= u(x)v'(x) + v(x)u'(x) \end{aligned}$$

2.13 Quotient Rule:

If u and v are differentiable at x and $v(x) \neq 0$, then the quotient u/v is differentiable at x and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

or

$$\left(\frac{u}{v} \right)' = \frac{vu' - uv'}{v^2}.$$

N.B. Due to the minus sign the order of u and v is very important on the right hand side.

2.14 Example

(i) Let

$$f(x) = \frac{(5x^2 + 1)}{(2x - 2)}.$$

Set $u(x) = 5x^2 + 1$ and $v(x) = 2x - 2$ so $f(x) = \frac{u(x)}{v(x)}$.

Use the quotient rule:

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Here $u'(x) = 10x$, $v'(x) = 2$.

By the quotient rule

$$f'(x) = \frac{10x(2x - 2) - (5x^2 + 1)2}{(2x - 2)^2} = \frac{10x^2 - 20x - 2}{(2x - 2)^2}.$$

(ii) Let $h(t) = \frac{(2t^3 + t)}{(t^2 - t)}$.

Set $u(t) = 2t^3 + t$ and $v(t) = t^2 - t$ so that $h(t) = \frac{u(t)}{v(t)}$.

Use the quotient rule:

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Here $u'(t) = 6t^2 + 1$, $v'(t) = 2t - 1$.

By the quotient rule

$$\begin{aligned} h'(t) &= \frac{(6t^2 + 1)(t^2 - t) - (2t^3 + t)(2t - 1)}{(t^2 - t)^2} \\ &= \frac{2t^4 - 4t^3 - t^2}{(t^2 - t)^2}. \end{aligned}$$

2.15 Negative powers

If n is a negative integer, ($n \leq -1$) and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Note $x^n = \frac{1}{x^{-n}}$ ($m = -n \geq 1$ is a positive integer).

Then

$$x^n = \frac{u}{v}$$

where $u = 1$ and $v = x^{-n} = x^m$.

Here $u' = 0$ and $v' = mx^{(m-1)}$.

By the Quotient rule

$$\begin{aligned}\frac{d}{dx}(x^n) &= \frac{x^m \times 0 - 1 \times (mx^{m-1})}{x^{2m}} = \frac{-mx^{m-1}}{x^{2m}} \\ &= -mx^{-m-1} = nx^{n-1}.\end{aligned}$$

2.16 Example

(i) $f(x) = x^{-5}$, $f'(x) = -5x^{-6}$, $f''(x) = 30x^{-7}$.

(ii) $g(t) = t^{-20}$, $g'(t) = -20t^{-21}$, $g''(t) = 420t^{-22}$.

(iii) $h(a) = a^5 - a^{-2}$, $h'(a) = 5a^4 + 2a^{-3}$, $h''(a) = 20a^3 - 6a^{-4}$.

(iv) Find an equation for the tangent to the curve $y = x + \frac{2}{x^2}$ at the point $(1, 3)$.

We first need to find the slope of the tangent at $(1, 3)$.

$$y' = 1 - \frac{4}{x^3}.$$

The slope at $x = 1$ is $y'(1) = 1 - 4 = -3$.

The line through $(1, 3)$ with slope -3 is

$$y = 3 - 3(x - 1) = -3x + 6.$$

Choosing which rule to use

(v)

Let

$$y = \frac{(x-1)(x^2-2x)}{x^4}.$$

Rather than using the quotient rule, expand the numerator and divide by x^4 :

$$y = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the sum and power rules:

$$y' = -x^{-2} + 6x^{-3} - 6x^{-4} = -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$