The Chain Rule

2.17 Example

(i) Let $y = 6x - 10 = 2(3x - 5)$.

Then $\dfrac{dy}{dx} = 6$.

We can also think of y as the composite of the functions $y = 2u$ and $u = 3x - 5$.

Then we have
$$\frac{dy}{du} = 2, \quad \frac{du}{dx} = 3.$$

So that
$$\frac{dy}{du}.\frac{du}{dx} = 2.3 = 6 = \frac{dy}{dx}.$$

If we think of the derivative as the rate of change we see that it makes sense that if $y = f(u)$ changes twice as fast as u and $u = g(x)$ changes three times as fast as x, then $y = f(g(x))$ changes six times as fast as x.

(ii) Let $y = (x^2 - 1)^2 = x^4 - 2x^2 + 1$.

Then $\dfrac{dy}{dx} = 4x^3 - 4x$.

We can also think of y as the composite of the functions $y = u^2$ and $u = x^2 - 1$.

Then we have
$$\frac{dy}{du} = 2u, \quad \frac{du}{dx} = 2x.$$

So that
$$\frac{dy}{du}.\frac{du}{dx} = 2u.2x = 2x(2x^2 - 2) = 4x^3 - 4x = \frac{dy}{dx}.$$

2.18 The Chain Rule
If g is differentiable at the point $u = f(x)$ and f is differentiable at x, then the composite function $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = f'(x).g'(f(x))$$

or if $y = g(u)$ and $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

Power Chain Rule

Suppose that $u = f(x)$ is a differentiable function of x, n is a non-zero integer and $y = g(u) = u^n$.

Then $\frac{dy}{du} = nu^{n-1}$ (by the power rule).

By the Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = nu^{n-1}\frac{du}{dx}.$$

or

$$y' = nu^{n-1}u'.$$

2.19 Example

(i) Let $y = (x^4 + 6x^2 - x)^{10}$.

Then $y = u^{10}$ where $u = x^4 + 6x^2 - x$.

We use the Power Chain Rule (with $n = 10$):

$$y' = 10u^9u'.$$

$u' = 4x^3 + 12x - 1$.

Therefore

$$y' = 10(4x^3 + 12x - 1)(x^4 + 6x^2 - x)^9.$$

(ii) Let $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$.

Then $y = u^4$ where $u = \frac{x^2}{8} + x - \frac{1}{x}$.

We use the Power Chain Rule (with $n = 4$):

$$y' = 4u^3u'.$$

$u' = \frac{x}{4} + 1 + \frac{1}{x^2}$.

Therefore

$$y' = 4\left(\frac{x}{4} + 1 + \frac{1}{x^2}\right)\left(\frac{x^2}{8} + x - \frac{1}{x}\right)^3.$$

(iii) Let

$$s = \frac{1}{(t^4 + 2t)^3}.$$

Then $s = u^{-3}$ where $u = t^4 + 2t$.

We use the Power Chain Rule (with $n = -3$):

$$s' = -3u^{-4}u'.$$

$u' = 4t^3 + 2$.

Therefore

$$s' = -3(4t^3 + 2)(t^4 + 2t)^{-4} = \frac{-12t^3 - 6}{(t^4 + 2t)^4}.$$

(iv) Let $y = \left(\frac{x^2 + 2x}{x - 5}\right)^8$.

Then $y = w^8$ where $w = \frac{x^2 + 2x}{x - 5}$.

We use the Power Chain Rule (with $n = 8$): $y' = 8w^7 w'$.

We need the quotient rule to find $w' : w = \frac{u}{v}$ where $u = x^2 + 2x, \quad v = x - 5$, and therefore

$$w' = \frac{(x-5)(2x+2) - (x^2 + 2x)}{(x-5)^2} = \frac{x^2 - 12x - 10}{(x-5)^2}.$$

So

$$\begin{aligned} y' &= \frac{8(x^2 - 12x - 10)}{(x-5)^2}\left(\frac{x^2 + 2x}{x - 5}\right)^7 \\ &= \frac{8(x^2 - 12x - 10)(x^2 + 2x)^7}{(x-5)^9}. \end{aligned}$$

2.20 Implicit Differentiation

(i) The equation of the circle centred at the origin of radius 1 is

$$x^2 + y^2 = 1.$$

If we want to find the tangent to the circle at the point $(0.6, -0.8)$, we can solve for y to get

$$y = \pm\sqrt{1 - x^2}.$$

We could then find $\dfrac{dy}{dx}$ in the relevant case.

Alternatively we have $y^2 = 1 - x^2$

Differentiate both sides with respect to x

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(1 - x^2)$$

Use the Power Chain rule

$$2y\frac{dy}{dx} = -2x.$$

Solve for $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = -\dfrac{2x}{2y} = -\dfrac{x}{y}.$

The tangent to the circle at the point $(0.6, -0.8)$ has slope $\dfrac{-0.6}{-0.8} = 0.75.$

The equation of the tangent is $y = 0.75x - 1.25.$

(ii) Find the tangent to the elliptic curve $y^2 = x^3 - x + 1.$ at the point $(-1, 1)$.

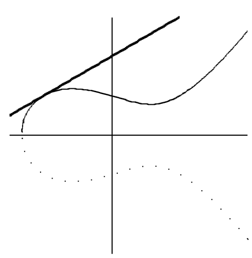

Figure 1: Graph of the elliptic curve $y^2 = x^3 - x + 1$.

We have $y^2 = x^3 - x + 1$

Differentiate both sides with respect to x

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - x + 1)$$

Use the Power Chain rule

$$2y\frac{dy}{dx} = 3x^2 - 1.$$

Solve for $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = \dfrac{3x^2 - 1}{2y}$.

The tangent to the curve at the point $(-1, 1)$ has slope $\dfrac{2}{2} = 1$. The equation of the tangent is $y = 1 + (x + 1) = x + 2$.

(iii)

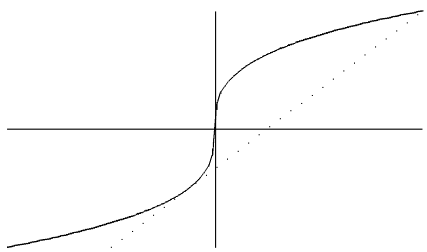

Figure 2: Tangent to $y = \sqrt[3]{x}$ ar $(-1, -1)$.

Let $y = \sqrt[3]{x}$. Find the tangent to the curve at the point $(-1, -1)$.

We have $y^3 = x$

Differentiate both sides with respect to x

$$\frac{d}{dx}(y^3) = \frac{d}{dx}(x)$$

Use the Power Chain rule $3y^2\dfrac{dy}{dx} = 1$.

Solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{3y^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

The tangent to the circle at the point $(-1, -1)$ has slope $\frac{1}{3}$. The equation of the tangent is $y = -1 + \frac{1}{3}(x+1) = \frac{1}{3}x - \frac{2}{3}$.

The derivative y' is defined for all $x \neq 0$.

2.21 Power rule for rational powers

If n is a non-zero rational number, then $y = x^n$ is differentiable at every interior point of the domain of $x^{(n-1)}$ and

$$\frac{d}{dx}(x^n) = nx^{(n-1)}.$$

So for example

$$y = x^{1/2} \quad \Longrightarrow y' = \frac{1}{2}x^{-1/2}.$$

We also have the Power Chain rule: If n is a rational number and u is a differentiable function of x, then $y = u^n$ is differentiable provided that $u \neq 0$ if $n < 1$ and

$$\frac{dy}{dx} = nu^{(n-1)}\frac{du}{dx}.$$

2.22 Example

Find y' when $y = \sqrt{x^3+1}$.

Here $y = u^{1/2}$ where $u = x^3 + 1$.

By the Power Chain Rule:

$$\frac{dy}{dx} = \frac{u^{-1/2}}{2}\frac{du}{dx} = \frac{3x^2}{2\sqrt{x^3+1}}.$$