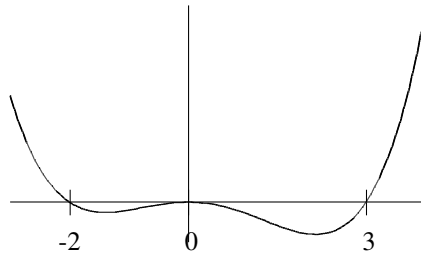


Maximum and minimum problems

Graphs



Last semester you saw how to draw the graph of a polynomial. We will use differentiation to add extra information:

Where are the *turning/extreme* points in the graph?

When is the function *increasing* or *decreasing*?

How can we determine the *shape* of the graph?

2.23 Definition

The function f is said to have a *local maximum* at c if

$$f(x) \leq f(c),$$

for all x in some open interval containing c .

The function f is said to have a *relative minimum* at c if

$$f(x) \geq f(c),$$

for all x in some open interval containing c .

A *local extremum* is either a *local maximum* or *local minimum*.

The function f is said to have an *absolute maximum* at c if

$$f(x) \leq f(c),$$

for all x in the domain of f .

The function f is said to have an *absolute minimum* at c if

$$f(x) \geq f(c),$$

for all x in the domain of f .

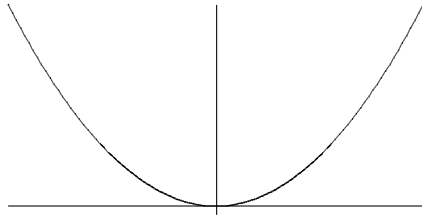


Figure 1: Graph of $f(x) = x^2$.

2.24 Example

(i)

The domain of $f(x) = x^2$ is \mathbb{R} .

f has an absolute minimum at $x = 0$.

Note $f'(x) = 2x$ and $f'(0) = 0$.

f has no local maximum.

(ii)

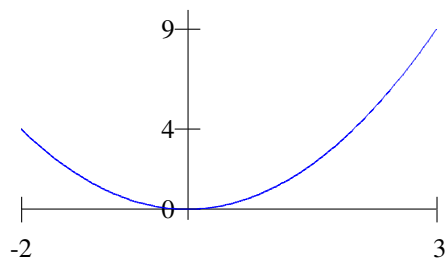


Figure 2: Graph of $f(x) = x^2$ for $x \in [-2, 3]$

The domain of $f(x) = x^2$ is $[-2, 3]$.

f has an absolute minimum at $x = 0$.

f has an absolute maximum at $x = 3$, $f(3) = 9$, $f'(3) = 6$.

f has a local maximum at $x = -2$, $f(-2) = 4$, $f'(-2) = -4$.

2.25 Theorem

If f has a local maximum or a local minimum at an interior point c of its domain and $f'(c)$ exists, then

$$f'(c) = 0.$$

2.26 Examples

(i)

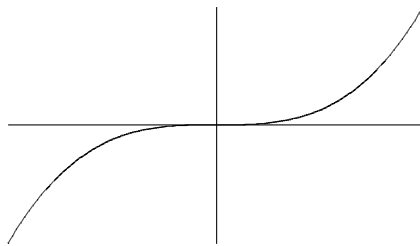


Figure 3: Graph of $f(x) = x^3$.

When $g(x) = x^3$, $g'(x) = 3x^2$ and $g'(0) = 0$ but g does not have a local extremum at $x = 0$. (ii)

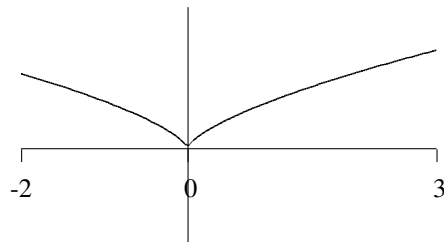


Figure 4: Graph of $f(x) = x^{2/3}$. for $x \in [-2, 3]$

Note $f'(x) = \frac{2}{3}x^{-1/3}$ and f' is not defined at $x = 0$. However we can see from the diagram that f has an absolute minimum at $x = 0$

2.27 Definition

A point in the domain of a function f at which $f' = 0$ or f' does not exist is a *critical point* of f .

The only places where a function f can have an extreme value are:

Interior points where $f' = 0$;

Interior points where f' is undefined;

Endpoints of the domain of f .

In the previous example:

(ii) $f(x) = x^{2/3}$ and $f'(x) = \frac{2}{3}x^{-1/3}$ for $x \in [-2, 3]$.

f' is not defined at $x = 0$ but is defined for all $x \neq 0$.

$f'(x) \neq 0$ if $x \neq 0$.

We check the value of f at its critical points and endpoints:

$$f(0) = 0, f(-2) = 4^{1/3}, f(3) = 9^{1/3}.$$

f has an **absolute minimum** at $x = 0$.

f has an **absolute maximum** at $x = 3$.

f has a **local maximum** at $x = -2$.

2.28 Example

(i) Find the extremum of $f(x) = x^3 + x^2 - 8x + 5$.

$$f'(x) = 3x^2 + 2x - 8 = (3x - 4)(x + 2).$$

$$f'(x) = 0 \Leftrightarrow (3x - 4) = 0 \text{ or } x + 2 = 0.$$

$$f'(x) = 0 \Leftrightarrow x = \frac{4}{3} \text{ or } x = -2.$$

$$f(4/3) = (4/3)^3 + (4/3)^2 - 8(4/3) + 5 = 1.52, f(-2) = -8 + 4 + 16 + 5 = 17.$$

$$\text{Note } \lim_{x \rightarrow -\infty} f(x) = -\infty, \lim_{x \rightarrow \infty} f(x) = \infty.$$

Therefore f has a **local minimum** at $x = 4/3$ and a **local maximum** at $x = -2$.

Further examples are done in lectures