The shape of a graph

2.29 Definition

The function f is said to be increasing on the interval I if

$$f(x_1) < f(x_2) \ \text{ whenever } \ x_1 < x_2$$

for $x_1, x_2 \in I$.

The function $y = x^2 + 1$ is increasing on the interval $[2, 6]$. Note $y' = 2x > 0$ on $[2, 6]$.

The function f is said to be decreasing on the interval I if

$$f(x_1) > f(x_2) \ \text{ whenever } \ x_1 < x_2.$$

The function $y = x^2+1$ is increasing on the interval $[-3, -1]$. Note $y' = 2x < 0$ on $[-3, -1]$.

2.30 Theorem
Suppose that f is differentiable on an open interval I. Then

f is increasing on I if $f'(x) > 0$ for all $x \in I$.

f is decreasing on I if $f'(x) < 0$ for all $x \in I$.

2.31 Example

(i) Let $f(x) = 2x^3 + 3x^2 - 36x + 27$.

Then $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$.

Note $f'(x) = 0$ when $x = -3$ or $x = 2$.

$f'(x) > 0$ as $x \to \pm\infty$.

At $x = -3$, f' crosses the x-axis and becomes *negative*.

At $x = 2$, f' crosses the x-axis and becomes *positive* again.

Therefore f is *increasing* on the intervals $(-\infty, -3)$ and $(2, \infty)$.

Also f is *decreasing* on the interval $(-3, 2)$.

(ii) Let $g(x) = \frac{x-1}{x+4}$.

Then, by the Quotient Rule:

$$g'(x) = \frac{(x+4).1 - (x-1).1}{(x+4)^2} = \frac{5}{(x+4)^2}$$

Neither g nor g' are defined when $x = -4$.

$(x+4)^2 > 0$, for all $x \neq -4$.

So $g'(x) > 0$ when $x \neq -4$.

Therefore g is *increasing* on the intervals $(-\infty, -4)$ and $(-4, \infty)$.

(iii) Let $h(x) = \frac{x^2 - 5x + 3}{x+1}$.

Done on board

(iv) Let $k(x) = \frac{x^2 - 3}{x^3}$.

Done on board

2.32 First Derivative Test for local Extrema

At a critical point $x = c$

f has a local minimum if f' changes from negative to positive at c.

f has a local maximum if f' changes from positive to negative at c.

f has no local extreme if f' has the same sign at both sides of c.

2.33 Second Derivative Test for local Extrema

If $f(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

If $f(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

2.34 Example

(i) Let $f(x) = 2x^3 + 3x^2 - 36x + 27$.

Then $f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2)$.

We saw that $f'(x) = 0$ when $x = -3$ or $x = 2$.

Now
$$f''(x) = 12x + 6 = 6(2x + 1).$$

$f''(-3) = 6(-6 + 1) = -30$ and so f has a local maximum at $x = -3$.

$f''(2) = 6(4 + 1) = 30$ and so f has a local minimum at $x = 2$.

(ii) Let $g(x) = \dfrac{x-1}{x+4}$.

Then
$$g'(x) = \frac{(x+4).1 - (x-1).1}{(x+4)^2} = \frac{5}{(x+4)^2}$$

In this case $g'(x) > 0$, for all $x \neq -4$ and g has no local extrema.

(iii) Let $h(x) = \dfrac{x^2 - 5x + 3}{x+1}$.

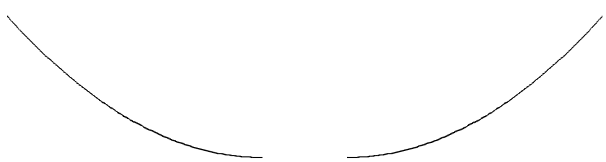
Done on board

(iv) Let $k(x) = \dfrac{x^2 - 3}{x^3}$.

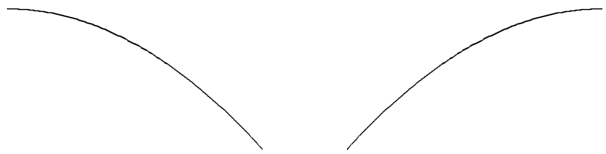
Done on board

2.35 Concavity

If a curve has the shape of a cup or part of a cup we say that it is concave up: If a



curve has the shape of a cap or part of a cap we say that it is concave down:



If

$$f''(x) > 0$$

on an open interval I, then f is concave up on I.

If

$$f''(x) < 0$$

on an open interval I, then f is concave down on I.