

2.36 Curve Sketching

In order to sketch the graph of a function f you need to do the following:

- (a) Determine the intervals (if there are any) on which f is positive and the intervals on which f is negative.
- (b) Determine the behaviour of $f(x)$ as $x \rightarrow \pm\infty$.
- (c) At each $x_0 \in \mathbb{R}$ where $f(x_0)$ is *not* defined, determine

$$\lim_{x \uparrow x_0} f(x) \text{ and } \lim_{x \downarrow x_0} f(x).$$

- (d) Determine the intervals (if there are any) on which f is increasing and the intervals on which f is decreasing.

2.37 Example

- (i) Sketch the graph of the function $f(x) = x^3 - 12x^2 = x^2(x - 12)$.
- (a) Note $f(x) = 0$ when $x = 0$ or $x = 12$. For $x \neq 0$, $x^2 > 0$. So that $f(x) > 0$ if $x - 12 > 0$ or $x > 12$ and $f(x) < 0$ if $x < 12$ and $x \neq 0$.

Therefore f is positive on the interval $(12, \infty)$ and f is negative on the intervals $(-\infty, 0)$ and $(0, 12)$.

- (b) The leading term is x^3 . So

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

$$\lim_{x \rightarrow -\infty} f(x) = -\infty.$$

- (c) $f(x)$ is defined for all $x \in \mathbb{R}$.
- (d) f is increasing on an interval I if $f'(x) > 0$ for all $x \in I$ and f is decreasing on an interval I if $f'(x) < 0$ for all $x \in I$.

$$f'(x) = 3x^2 - 24x = 3x(x - 8).$$

$$\lim_{x \rightarrow \pm\infty} f'(x) = \infty \text{ and } f'(x) = 0 \text{ when } x = 0 \text{ and } x = 8.$$

f' is positive and so f is increasing on the intervals $(-\infty, 0)$ and $(8, \infty)$.

f' is negative and f is decreasing on the interval $(0, 8)$.

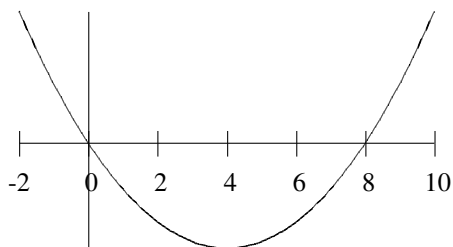


Figure 1: Graph of $f'(x) = 3x^2 - 24x$.

Alternatively you can make a table to determine where f' is positive and negative:

	$x < 0$	$0 < x < 8$	$8 < x$
$3x$	-	+	+
$x - 8$	-	-	+
$f'(x)$	+	-	+

From the table we see that f' is positive and so f is increasing on the intervals $(-\infty, 0)$ and $(8, \infty)$.

f' is negative and so f is decreasing on the interval $(0, 8)$.

Graph done on board.

(ii) Sketch the graph of the function $f(x) = \frac{x}{x^2 + 1}$.

(a) Note $f(x) = 0$ when $x = 0$.

$$f(x) > 0 \Leftrightarrow x(x^2 + 1) > 0 \Leftrightarrow x > 0.$$

Therefore f is positive on the interval $(0, \infty)$ and f is negative on the interval $(-\infty, 0)$.

(b) The highest power of x in the denominator is x^2 so

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x/x^2}{x^2/x^2 + 1/x^2} = \lim_{x \rightarrow \infty} \frac{1/x}{1 + 1/x^2} = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x/x^2}{x^2/x^2 + 1/x^2} = \lim_{x \rightarrow -\infty} \frac{1/x}{1 + 1/x^2} = 0.$$

The line $y = 0$ is a horizontal asymptote of the graph.

(c) $f(x)$ is defined for all $x \in \mathbb{R}$.

(d) By the Quotient rule $f'(x) = \frac{1-x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$.

f' is positive when $1-x^2 > 0$ and so f is increasing on the interval $(-1, 1)$.

f' is negative and f is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$.

Note $f(1) = 1/2$, $f(-1) = -1/2$.

Graph done on board.

(iii) Sketch the graph of the function $f(x) = \frac{x+3}{x+2}$.

(iv) Sketch the graph of the function $f(x) = -\frac{8}{x^2-4}$.

(a)

$$\begin{aligned} f(x) > 0 &\Leftrightarrow -8(x^2-4) > 0 \\ &\Leftrightarrow x^2-4 = (x-2)(x+2) < 0 \Leftrightarrow -2 < x < 2. \end{aligned}$$

Therefore f is positive on the interval $(-2, 2)$ and f is negative on the intervals $(-\infty, -2)$ and $(2, \infty)$.

(b)

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} -\frac{8}{x^2-4} = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} -\frac{8}{x^2-4} = 0.$$

The line $y = 0$ (x-axis) is a horizontal asymptote of the graph.

(c) $f(x)$ is not defined when $x = \pm 2$. As $x \uparrow \pm 2$ and $x \downarrow \pm 2$,

$$f(x) \rightarrow \pm\infty$$

(the sign depends on whether f is positive or negative in that region).

If $x < -2$, then $f(x) < 0$ and so

$$\lim_{x \uparrow -2} f(x) = -\infty.$$

If $-2 < x < 2$, then $f(x) > 0$ and so

$$\lim_{x \downarrow -2} f(x) = \infty = \lim_{x \uparrow 2} f(x).$$

If $2 < x$, then $f(x) < 0$ and so

$$\lim_{x \downarrow 2} f(x) = -\infty.$$

The lines $x = -2$ and $x = 2$ are vertical asymptotes of the graph.

(d) By the Quotient rule

$$f'(x) = \frac{16x}{(x^2 - 4)^2}.$$

f' is negative, for all $x < 0, x \neq -2$. So f is decreasing on the intervals $(-\infty, -2)$ and $(-2, 0)$.

f' is positive, for all $x > 0, x \neq 2$. So f is increasing on the intervals $(0, 2)$ and $(2, \infty)$.

Graph done on board.

(v) Sketch the graph of the function $f(x) = \frac{3x+1}{x^3-1}$.

(a)

$$\begin{aligned} f(x) < 0 &\Leftrightarrow (3x+1)(x^3-1) < 0 \\ &\Leftrightarrow -1/3 < x < 1. \end{aligned}$$

Therefore f is positive on the intervals $(-\infty, -1/3)$ and $(1, \infty)$ and f is negative on the interval $(-1/3, 1)$.

(b)

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{3x+1}{x^3-1} = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{3x+1}{x^3-1} = 0.$$

The line $y = 0$ (x-axis) is a horizontal asymptote of the graph.

(c) $f(x)$ is not defined when $x = 1$. As $x \uparrow 1$ and $x \downarrow 1$,

$$f(x) \rightarrow \pm\infty$$

(the sign depends on whether f is positive or negative in that region).

If $-1/3 < x < 1$, then $f(x) < 0$ and so

$$\lim_{x \uparrow 1} f(x) = -\infty.$$

If $1 < x$, then $f(x) > 0$ and so

$$\lim_{x \downarrow 1} f(x) = \infty.$$

The line $x = 1$ is a vertical asymptote of the graph.

(d) By the Quotient rule

$$f'(x) = \frac{-3(2x^3 + x^2 + 1)}{(x^3 - 1)^2} = \frac{-3(x+1)(2x^2 - x + 1)}{(x^3 - 1)^2}.$$

f' is negative, for all $x > -1$, $x \neq 1$. So f is decreasing on the intervals $(-1, 1)$ and $(1, \infty)$.

f' is positive, for all $x < -1$. So f is increasing on the interval $(-\infty, -1)$.

Graph done on board.