Functions 1

A function is a special kind of binary relation. Specifically, a function " f " from a set X to a set Y is a relation from X to Y (i.e. a subset of $X \times Y$) satisfying the property that for each $x \in X$ there is exactly one $y \in Y$ (denoted by $f(x)$) such that x is related to $f(x)$. That is, $(x,\ f(x))$ is in the relation.

Diagram 1.

Despite the fact that a function is indeed a relation, we generally do not think of it in this way. Rather, we think of a function and use it as explained in the following:

Definition (A Function) : A function " f " from a set X to a set Y is a " rule " which to each $x \in X$ associates a **unique** $f(x) \in Y$, called the image of x under f or the value of f at x.

Notation : To denote a function " f " from X to Y we usually write:

$$f : X \to Y : x \mapsto f(x).$$

Diagram 2.

Warning : When we use the term " a **unique** $f(x) \in Y$ " in the definition above all we mean is the once the input x is specified there is only one output $f(x)$. We do **not** mean that two different inputs, x_1 and x_2 say, must result in two different outputs $f(x_1)$ and $f(x_2)$. In fact, it may be the case that every $x \in X$ results in the same output. That is, $f(x) = c$ for all $x \in X$. Such functions are called constant functions

Diagram 3.

Terminology : We use the following terminology:

(i) The set X is called the **domain** of f, or informally the set of **inputs** of f.

(ii) The set Y is called the **codomain** of f, or informally the set containing **outputs** from f.

(iii) The symbol $x \mapsto f(x)$ corresponds to the **rule** by which x is associated to $f(x)$.

Remark : In computing you have exactly these kind of functions or procedures. In C and $C++$ they are actually called functions. Each such function has a name, here " f ". The first thing you must do when writing such a function is to specify the set of inputs (i.e. the set X). Next you must specify the output set (i.e. the set Y) and finally you must write the computer programme itself, that is the rule $x \mapsto f(x)$ which specifies (usually in a step by step way) exactly how the output $f(x)$ is to be obtained from the input x.

Example [1] : In school you might have considered such things as:

(i) The function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto 2x - 1\,.$$

Thus $f(x) = 2x - 1$ for which in school you might have written $y = 2x - 1$.

(ii) The function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2\,.$$

Thus $f(x) = x^2$ for which in school you might have written $y = x^2$.

(iii) The function

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{3x - 4}{x^2 + 1}\,.$$

Thus $f(x) = \dfrac{3x - 4}{x^2 + 1}$ for which in school you might have written $y = \dfrac{3x - 4}{x^2 + 1}$.

Of course functions can be much more complicated than those that you saw in school. Consider the following:

Example [2] : In the following the sets X, Y and Z will be sets of record files of specified record types in some database, see the notes in the file " Relations 3 ".

(i) The function $\text{Project}_{\text{attributes}}$ given by

$$\text{Project}_{\text{attributes}} : X \longrightarrow Y : \boldsymbol{\mathcal{F}} \mapsto \text{Project}\,[\,\boldsymbol{\mathcal{F}},\ \text{attributes}].$$

(ii) The function $\text{Select}_{\text{criterion}}$ given by

$$\text{Select}_{\text{criterion}} : X \longrightarrow X : \boldsymbol{\mathcal{F}} \mapsto \text{Select}\,[\,\boldsymbol{\mathcal{F}},\ \text{criterion}].$$

(iii) The function Join given by

$$\text{Join} : X \times Y \longrightarrow Z : (\,\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}\,) \mapsto \text{Join}\,[\,\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}\,].$$

Definition (The Graph of a Function) : Given a function

$$f : X \to Y : x \mapsto f(x),$$

its graph (denoted by **graph**(f)) is by definition given by

$$\text{graph}(\mathbf{f}) \;=\; \{\,(\,x,\; f(x)\,)\;\mid\; x \in X\,\}$$

Note: $\text{graph}(f) \subset X \times Y$.

Note: Additional examples and pictures will be given in class.