Functions 2

Interval Notation : The following is a standard notation for intervals of real numbers which we will use frequently throughout the course. Here a and b will be real numbers or $\pm\infty$.

(i) $[\,a,\,b\,] = \{\,x \in \mathbb{R} \mid a \leq x \leq b\,\}$.

(ii) $[\,a,\,b\,) = \{\,x \in \mathbb{R} \mid a \leq x < b\,\}$.

(iii) $(\,a,\,b\,] = \{\,x \in \mathbb{R} \mid a < x \leq b\,\}$.

(iv) $(\,a,\,b\,) = \{\,x \in \mathbb{R} \mid a < x < b\,\}$.

Natural domain of definition of functions : Here we consider real-valued functions of a real variable, that is functions of the type

$$f : X \longrightarrow \mathbb{R} : x \longmapsto f(x) \quad \text{where} \quad X \subseteq \mathbb{R}.$$

By definition, the **natural domain of definition of f** is the largest subset $X \subseteq \mathbb{R}$ such that $f(x)$ is defined for all $x \in X$.

Example [1] : In each of the following $X \subseteq \mathbb{R}$.

(i) The function $f : X \longrightarrow \mathbb{R} : x \longmapsto 2x - 1$ has natural domain of definition

$$X = \mathbb{R}$$

(ii) The function $f : X \longrightarrow \mathbb{R} : x \longmapsto \dfrac{1}{x+3}$ has natural domain of definition

$$X = \mathbb{R} \setminus \{-3\}.$$

(iii) The function $f : X \longrightarrow \mathbb{R} : x \longmapsto \dfrac{1}{(x+3)(x-5)}$ has natural domain of definition

$$X = \mathbb{R} \setminus \{-3,\,5\}.$$

(iv) The function $f : X \to \mathbb{R} : x \mapsto \sqrt{x-1}$ has natural domain of definition

$$X = [\,1,\, \infty\,).$$

Definition (Composition of functions) : Suppose that we are given two functions

$$g : W \to X : w \mapsto g(w) \quad \text{and} \quad f : X \to Y : x \mapsto f(x),$$

then we can construct a new function called the composition of f **after** g, which is denoted by $(f \circ g)$, and is defined by:

$$(f \circ g) : W \to Y : w \mapsto f(g(w))$$

Diagram 1.

Example [2] :

(i) The composition function $(f \circ g)$ of the functions

$$g : \mathbb{R} \to \mathbb{R} : w \mapsto (3w+1) \quad \text{and} \quad f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2\,.$$

is

$$(f \circ g) : \mathbb{R} \to \mathbb{R} : w \mapsto f(g(w)) \;=\; (g(w))^2 \;=\; (3w+1)^2$$

(ii) The composition function $(f \circ g)$ of the functions

$$g : \mathbb{R} \to [\,1,\, \infty\,) : w \mapsto (w^2+1) \quad \text{and} \quad f : (\,0,\, \infty\,) \to \mathbb{R} : x \mapsto \frac{x-3}{\sqrt{x}+5}\,.$$

is

$$(f \circ g) : \mathbb{R} \to \mathbb{R} : w \mapsto (f(g(w)) \;=\; \frac{g(w)-3}{\sqrt{g(w)}+5} \;=\; \frac{(w^2+1)-3}{\sqrt{(w^2+1)}+5} \;=\; \frac{w^2-2}{\sqrt{w^2+1}+5}$$

Example [3] : In computing you can have the composition of functions by calling one function into another, that is, first perform one and use the result in another. For example, in relational databases, you could compose functions:

$$\text{Project}_{\text{attributes}} : X \longrightarrow Y : \boldsymbol{\mathcal{F}} \mapsto \text{Project}\,[\,\boldsymbol{\mathcal{F}},\ \text{attributes}]$$

and

$$\text{Select}_{\text{criterion}} : X \longrightarrow X : \boldsymbol{\mathcal{F}} \mapsto \text{Select}\,[\,\boldsymbol{\mathcal{F}},\ \text{criterion}]$$

to get $(\text{Project}_{\text{attributes}} \circ \text{Select}_{\text{criterion}})$.

Associativity of the Composition of Functions : Suppose that we are given the following three functions

$$h : V \to W : v \mapsto h(v), \quad g : W \to X : w \mapsto g(w) \quad \text{and} \quad f : X \to Y : x \mapsto f(x),$$

then we can form a composition of them in two ways:

Diagram 2. $f \circ [g \circ h]$

and

Diagram 3. $[f \circ g] \circ h$

Now, for all $v \in V$, we have

$$\begin{aligned} (\, f \circ [g \circ h]\,)(v) &= f(\,[g \circ h](v)\,) \\ &= f(\,g(h(v))\,) \\ &= [f \circ g](h(v)) \\ &= (\,[f \circ g] \circ h\,)(v) \end{aligned}$$

so that

$$(\, f \circ [g \circ h]\,) = (\,[f \circ g] \circ h\,)\,.$$

This identity is referred to by saying that **the composition of functions is associative**.

Definition (The Identity Function on a Set) : For any set X we define the **the identity function on X** to be the function:

$$id_X : X \to X : x \mapsto x\,.$$

That is $id_X(x) = x$ for all $x \in X$, or in other words, the identity function on X leaves everything in X alone.

Diagram 4.

Though it might seem that identity functions are quite useless (since they "do nothing"). However, we will find that they play a very important role in what follows. This role is somewhat analogous to the role of "1" when multiplying real numbers.

Definition (Left Inverses) : Given a function $f : X \longrightarrow Y : x \mapsto f(x)$, then a function

$$g : Y \longrightarrow X : y \mapsto g(y)$$

is called a (left inverse) of f if and only if $g(f(x)) = x$ for all $x \in X$.

Diagram 5.

Or, in other words: $\boxed{g \text{ is a left inverse of } f} \iff \boxed{(g \circ f) \ = \ id_X}$.

Note: If you think of the function f as an **encoder**, then a (left inverse) of f is a **decoder**.

Definition (Right Inverses) : Similar to the above, given a function $f : X \longrightarrow Y : x \mapsto f(x)$, then a function

$$h : Y \longrightarrow X : y \mapsto h(y)$$

is called a (right inverse) of f if and only if $f(h(y)) = y$ for all $y \in Y$.

Diagram 6.

Or, in other words: $\boxed{h \text{ is a right inverse of } f} \iff \boxed{(f \circ h) \ = \ id_Y}$.

Important Observations : Start with a function $f : X \to Y : x \mapsto f(x)$ and consider the following:

Observation 1 : If f has a left inverse g, then

$$\boxed{f(x_1) = f(x_2)} \implies \boxed{g(f(x_1)) = g(f(x_2))}$$

$$\implies \boxed{x_1 = x_2}$$

That is, if f has a left inverse then distinct x's are mapped to distinct y's by f. Such functions are said to be **one-to-one** or **injective**.

Example [4]: The function

$$f : \mathbb{R} \to [0, \infty) : x \mapsto x^2$$

is **not injective** [for example $f(-2) = 4 = f(2)$] and, therefore, has no left inverse. However, this f has several **right inverses**. Here's one:

$$h : [0, \infty) \to \mathbb{R} : y \mapsto \sqrt{y}\,.$$

Observation 2 : If f has a right inverse h, then

$$\boxed{y \in Y} \implies \boxed{y = f(h(y))}$$

That is, if f has a right inverse then every $y \in Y$ is f(of some x), in fact, one such x is $x = h(y)$. Such functions are said to be **onto** or **surjective**.

Example [5]: The function

$$f : [0, \infty) \to \mathbb{R} : x \mapsto x^2$$

is **not surjective** [for example $-2 \neq f(\text{anything})$] and, therefore, has no right inverse. However, this f has several **left inverses**. Here's one:

$$g : \mathbb{R} \to [0, \infty) : y \mapsto \begin{cases} 0 & \text{if} \quad y < 0\,, \\ \sqrt{y} & \text{if} \quad y \geq 0\,. \end{cases}$$

Remark 1: The examples given in **Observation 1** and **Observation 2** above illustrate the fact that, for a function $f : X \to Y : x \mapsto f(x)$, any one of the following may be true:

(i) f may have no left inverse.

(ii) f may have exactly one left inverse.

(iii) f may have several left inverses.

(iv) f may have no right inverse.

(v) f may have exactly one right inverse.

(vi) f may have several right inverses.

Thus, the situation regarding the existence, or otherwise, of left or right inverses is (in general) quite complicated. Never-the-less, there is one very special and important case which we now address in

Proposition 1: If a function $f : X \to Y : x \mapsto f(x)$ has **both** a left inverse g and a right inverse h, then $g = h$.

Proof :

$$\begin{aligned} g &= g \circ id_Y \\ &= g \circ (f \circ h) && \text{Since } h \text{ is a right inverse of } f. \\ &= (g \circ f) \circ h && \text{Since composition of functions is associative.} \\ &= id_X \circ h && \text{Since } g \text{ is a left inverse of } f. \\ &= h\,. \end{aligned}$$

Corollary 1: If a function $f : X \to Y : x \mapsto f(x)$ has **both** a left inverse g and a right inverse h, then it has only one of each and they are equal

Proof : Fix a right inverse h. If g_1 and g_2 are left inverse of f, then (by Proposition 1) $g_1 = h$ and $g_2 = h$. Thus $g_1 = g_2$ and, therefore, there is only one left inverse - let's call it g. Similarly there is only one right inverse - call it h and, finally (again by the Proposition 1) $g = h$.

Definition (The Inverses of a Function) : In the special case given in the statement of Corollary 1 above, where a function $f : X \to Y : x \mapsto f(x)$ has both a left inverse and a right inverse, then this **unique** left (and simultaneously right) inverse of f is called simply **the inverse of f** and is denoted by f^{-1}.

Thus, **the inverse of** $f : X \to Y : x \mapsto f(x)$, when it exists, is the function

$$f^{-1} : Y \to X : y \mapsto f^{-1}(y)$$

which is uniquely determined by the conditions that:

$$\boxed{f^{-1}(f(x)) = x \quad \text{for all } x \in X} \quad \text{and} \quad \boxed{f(f^{-1}(y)) = y \quad \text{for all } y \in Y} \; .$$

Remark 2: If follows from **Observation 1** and **Observation 2**, given prior to Remark 1 above, that for a function $f : X \to Y : x \mapsto f(x)$

$$\boxed{\text{the existence of } f^{-1}} \implies \boxed{f \text{ is } \textbf{injective} \text{ and } \textbf{surjective}} .$$

The **converse** of this last statement is also true. This we now prove in

Proposition 2: In the case of a function $f : X \to Y : x \mapsto f(x)$ we have that

$$\boxed{f \text{ being both } \textbf{injective} \text{ and } \textbf{surjective}} \implies \boxed{\text{the existence of } f^{-1}} .$$

Proof : To prove this we must, for each $y_0 \in Y$, determine $f^{-1}(y_0)$. There are two steps:

Step 1: Since f is **surjective** we know that for each $y_0 \in Y$ there is some $x_0 \in X$ such that $f(x_0) = y_0$.

Step 2: Since f is **injective** we know that the $x_0 \in X$ obtained in Step 1 is unique. We define $f^{-1}(y_0)$ to be this unique $x_0 \in X$.

[QED]

Putting together the statements in Remark 2 and Proposition 2, we have proved

Corollary 2: For any function $f : X \to Y : x \mapsto f(x)$

$$\boxed{f^{-1} \text{ exists}} \iff \boxed{f \text{ is both } \textbf{injective} \text{ and } \textbf{surjective}} .$$

Terminology: A function which is both injective and surjective is called **bijective**, thus

$$\boxed{f^{-1} \text{ exists}} \iff \boxed{f \text{ is } \textbf{bijective}} .$$

How to Find The Inverses of a Function : When it exists, we can determine $f^{-1}(y)$ by observing that

$$\boxed{y = f(x)} \iff \boxed{f^{-1}(y) = x}.$$

That is, to find the " formula for $f^{-1}(y)$ " all we have to do is to solve the equation $y = f(x)$ for x in terms of y.

Example [6]: Given that the function

$$f : [\,0,\,\infty) \to [\,0,\,1) : x \mapsto f(x) = \frac{x}{x+1}$$

is **bijective** find f^{-1}.

Solution: By definition

$$f^{-1} : [\,0,\,1) \to [\,0,\,\infty) : y \mapsto f^{-1}(y),$$

so all we have to do is to find the formula for $f^{-1}(y)$. This we do by solving the equation $y = f(x)$ for x in terms of y. Proceed as follows:

$$\begin{array}{rccl}
 & y & = & f(x) \\
\Longrightarrow & y & = & \dfrac{x}{x+1} \\
\Longrightarrow & y\,(x+1) & = & x \\
\Longrightarrow & y\,x + y - x & = & 0 \\
\Longrightarrow & y\,x - x + y & = & 0 \\
\Longrightarrow & (y - 1)\,x + y & = & 0 \\
\Longrightarrow & (y - 1)\,x & = & -y \\
\Longrightarrow & x & = & \dfrac{-y}{y - 1} \\
\Longrightarrow & x = f^{-1}(y) & = & \dfrac{y}{1 - y}\,.
\end{array}$$

Thus,

$$f^{-1} : [\,0,\,1) \to [\,0,\,\infty) : y \mapsto f^{-1}(y) \;=\; \frac{y}{1-y}\,.$$

Note: Additional examples and pictures will be given in class.