The Principle of Mathematical Induction

The Context : Suppose that corresponding to each $n \in \mathbb{N}$ there is associated a proposition $\wp(n)$. That is, we have an infinite number of propositions:

$$\wp(1),\ \wp(2),\ \wp(3),\ \ldots\ \wp(n),\ \ldots$$

We will be interested in the case where all of these propositions are true and it is by using the principle of mathematical induction (see below) that we will prove this. However, until we actually prove that these propositions are in fact true we should refer to them only as hypotheses. When we do this we will refer to $\wp(n)$ as the induction hypothesis for the integer n.

Example [1] : For each $n \in \mathbb{N}$ let $\wp(n)$ be the proposition that:

$$1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2 .$$

Example [2] : For each $n \in \mathbb{N}$ let $\wp(n)$ be the proposition that:

$$(5^n - 1) \quad \text{is divisible by } 4.$$

Example [3] : For each fixed $x \in [-1, \infty)$ and for each $n \in \mathbb{N}$ let $\wp(n)$ be the proposition that:

$$(1 + x)^n \ \geq\ 1 + n\,x\,.$$

Example [4] : For each $n \in \mathbb{N}$ let $\wp(n)$ be the proposition that:

$$\frac{1}{1(1+2)} + \frac{1}{2(2+2)} + \frac{1}{3(3+2)} + \cdots + \frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right].$$

The Principle of Mathematical Induction : Suppose that we have an infinite number of propositions:

$$\wp(1),\ \wp(2),\ \wp(3),\ \ldots\ \wp(n),\ \ldots$$

(one corresponding to each $n \in \mathbb{N}$) which satisfy the following two conditions:

Condition 1 : $\wp(1)$ is true

Condition 2 : For each $k \in \mathbb{N}$ $\boxed{\text{the truth of } \wp(k)} \implies \boxed{\text{the truth of } \wp(k+1)}$,

then the principle of mathematical induction states that the proposition $\wp(n)$ is true for all $n \in \mathbb{N}$.

The reasoning behind the principle of induction : You can think of this principle as being analogues to an infinite ladder where the truth of $\wp(n)$ corresponds to being able to reach the n^{th}-rung of this ladder. Thus, if the following two conditions hold:

Condition 1 : You can reach the 1^{st}-rung.

Condition 2 : For each $k \in \mathbb{N}$, $\boxed{\begin{array}{c}\text{being on the}\\ k^{\text{th}}\text{-rung}\end{array}} \implies \boxed{\begin{array}{c}\text{you can reach the}\\ \text{next rung, i.e. you can}\\ \text{reach the } (k+1)^{\text{st}}\text{-rung}\end{array}}$,

then you can reach any rung on the ladder. That is, you can reach the n^{th}-rung for all $n \in \mathbb{N}$. This is the case simply because (read the diagram below from the bottom up):

$\vdots$		$\vdots$
$\Uparrow$	By condition 2	$\Uparrow$
You can reach the $(k+1)^{\text{st}}$-rung		$\wp(k+1)$ is true.
$\Uparrow$	By condition 2	$\Uparrow$
You can reach the k^{th}-rung		$\wp(k)$ is true.
$\Uparrow$	By condition 2	$\Uparrow$
$\vdots$		$\vdots$
$\Uparrow$	By condition 2	$\Uparrow$
You can reach the 4^{th}-rung		$\wp(4)$ is true.
$\Uparrow$	By condition 2	$\Uparrow$
You can reach the 3^{rd}-rung		$\wp(3)$ is true.
$\Uparrow$	By condition 2	$\Uparrow$
You can reach the 2^{nd}-rung		$\wp(2)$ is true.
$\Uparrow$	By condition 2	$\Uparrow$
Having reached the 1^{st}-rung	By condition 1	$\wp(1)$ is true.

We now apply the principle of induction to establish the truth of the propositions given in Examples [2] to [4] given above.

Example [1] : Prove by induction on $n \in \mathbb{N}$ that:

$$1+3+5+7+\ldots+(2n-1) = n^2\,. \quad (\text{ This is proposition } \wp(n).)$$

That is, the sum of the first n odd integers is n^2.

Proof :

Step 1 : Prove that $\wp(1)$ is true: With $n = 1$,

" left-hand-side " $= 1$ and

" right-hand-side " $= 1^2 = 1$.

Thus, $\wp(1)$ is true.

Step 2: Show that for all $k \in \mathbb{N}$, $\boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k)\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k+1)\end{array}}$.

To show that $\wp(k+1)$ is true, we proceed by looking at the sum of the first $(k+1)$ odd integers:

$$\begin{aligned} 1+3+5+\ldots+(2[k+1]-1) &= \{\,\text{sum of the first } k \text{ terms}\,\} + (k+1)^{\text{st}} \text{ term} \\ &= \{\,1+3+5+\ldots+(2k-1)\,\} + (2[k+1]-1) \\ &= \{\,k^2\,\} + (2[k+1]-1) \quad \left\{ \begin{array}{c}\text{By the induction}\\ \text{hypothesis that}\\ \wp(k) \text{ is true.}\end{array}\right. \\ &= k^2 + 2k + 1 \\ &= (k+1)^2\,. \end{aligned}$$

That is, $\wp(k+1)$ is also true.

[QED]

Example [2] : Prove by induction on $n \in \mathbb{N}$ that:

$$(5^n - 1) \quad \text{is divisible by } 4. \qquad (\text{ This is proposition } \wp(n).)$$

Proof :

Step 1 : Prove that $\wp(1)$ is true:

With $n = 1$; $(5^n - 1) = (5 - 1) = 4$, which is certainly divisible by 4 so that $\wp(1)$ is true.

Step 2: Show that for all $k \in \mathbb{N}$, $\boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k)\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k+1)\end{array}}$.

To show that $\wp(k+1)$ is true, we proceed by looking at $(5^{(k+1)} - 1)$. Observe that:

$$\begin{aligned}
5^{(k+1)} - 1 &= 5^{(k+1)} - 5^k + 5^k - 1 \\
&= \{5^{(k+1)} - 5^k\} + (5^k - 1) \\
&= \{5^k(5 - 1)\} + (5^k - 1) \\
&= \{4 \times 5^k\} + (5^k - 1) \\
&= \{4 \times 5^k\} + (4 \times \text{ some integer } m_1) \quad \left\{ \begin{array}{l} \text{By the induction} \\ \text{hypothesis that} \\ \wp(k) \text{ is true.} \end{array} \right. \\
&= 4 \times [5^k + \text{ some integer } m_1] \\
&= 4 \times [\text{ some integer } m].
\end{aligned}$$

That is, $(5^{(k+1)} - 1)$ is divisible by 4 so that $\wp(k+1)$ is also true.

[QED]

Example [3] : For each fixed $x \in [-1, \infty)$, prove by induction on $n \in \mathbb{N}$ that:

$$(1+x)^n \;\geq\; 1 + n\,x\,. \qquad (\text{ This is proposition } \wp(n).)$$

Proof :

Step 1 : Prove that $\wp(1)$ is true: With $n = 1$,

" left-hand-side " $= 1 + x$ and

" right-hand-side " $= 1 + x\,.$

Thus, $\wp(1)$ is true.

Step 2: Show that for all $k \in \mathbb{N}$, $\boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k)\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{the truth}\\ \text{of } \wp(k+1)\end{array}}$.

To show that $\wp(k+1)$ is true, we proceed by looking at $(1+x)^{(k+1)}$. Observe that:

$$\begin{aligned}
(1+x)^{(k+1)} &= (1+x)^k(1+x) \\
&\geq (1+k\,x)(1+x) \quad \left\{ \begin{array}{c}\text{By using the fact}\\ \text{that } \quad (1+x) > 0\\ \text{together with the}\\ \text{induction hypothesis}\\ \text{that } \wp(k) \text{ is true.}\end{array}\right. \\
&= 1 + k\,x + x + k\,x^2 \\
&\geq 1 + k\,x + x\,, \qquad \text{since } k\,x^2 \geq 0\,. \\
&\geq 1 + (k+1)\,x\,.
\end{aligned}$$

That is, $(1+x)^{(k+1)} \geq 1 + (k+1)\,x$ so that $\wp(k+1)$ is also true.

[QED]

Example [4] : Prove by induction on $n \in \mathbb{N}$ that:

$$\frac{1}{1(1+2)} + \frac{1}{2(2+2)} + \frac{1}{3(3+2)} + \cdots + \frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right].$$

For convenience we will use the shorter (summation) notation and write

$$\sum_{j=1}^{n} \frac{1}{j(j+2)} \quad \text{to mean} \quad \frac{1}{1(1+2)} + \frac{1}{2(2+2)} + \frac{1}{3(3+2)} + \cdots + \frac{1}{n(n+2)}.$$

Thus we wish to prove by induction on $n \in \mathbb{N}$ that

$$\sum_{j=1}^{n} \frac{1}{j(j+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right]. \quad \left\{ \begin{array}{l} \text{This is the} \\ \text{proposition} \\ \wp(n). \end{array} \right.$$

Proof :

Step 1 : Prove that $\wp(1)$ is true: With $n = 1$,

$$\text{“ left-hand-side ”} = \frac{1}{1(1+2)} = \frac{1}{3} \quad \text{and}$$

$$\text{“ right-hand-side ”} = \frac{1}{2}\left[\frac{3}{2} - \frac{2\,(1)+3}{(1+1)(1+2)}\right] = \frac{1}{2}\left[\frac{3}{2} - \frac{5}{6}\right] = \frac{1}{2}\left[\frac{9-5}{6}\right] = \frac{1}{3}.$$

Thus, $\wp(1)$ is true.

Step 2: Show that for all $k \in \mathbb{N}$, $\boxed{\begin{array}{c}\text{the truth} \\ \text{of } \wp(k)\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{the truth} \\ \text{of } \wp(k+1)\end{array}}$.

To show that $\wp(k+1)$ is true, we proceed by looking at the sum of the first $(k+1)$-terms, which we write as:

$$\boxed{\begin{array}{c}\text{the sum of the} \\ \text{first } (k+1) \text{ terms}\end{array}} = \boxed{\begin{array}{c}\text{the sum of the} \\ \text{first } k \text{ terms}\end{array}} + \boxed{\begin{array}{c}\text{the } (k+1)^{\text{st}} \\ \text{term}\end{array}}.$$

That is;

$$
\begin{aligned}
\sum_{j=1}^{k+1} \frac{1}{j(j+2)} &= \sum_{j=1}^{k} \frac{1}{j(j+2)} + \frac{1}{[k+1]([k+1]+2)} \\
\left.\begin{array}{c}\text{By the induction}\\ \text{hypothesis that}\\ \wp(k)\ \text{is true.}\end{array}\right\} \quad &= \frac{1}{2}\left[\frac{3}{2} - \frac{2k+3}{(k+1)(k+2)}\right] + \frac{1}{(k+1)(k+3)} \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\left[\frac{2k+3}{(k+1)(k+2)} - \frac{2}{(k+1)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\,\frac{1}{(k+1)}\left[\frac{2k+3}{(k+2)} - \frac{2}{(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\,\frac{1}{(k+1)}\left[\frac{(2k+3)(k+3)-2(k+2)}{(k+2)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\,\frac{1}{(k+1)}\left[\frac{2k^2+9k+9-2k-4}{(k+2)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\,\frac{1}{(k+1)}\left[\frac{2k^2+7k+5}{(k+2)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\,\frac{1}{(k+1)}\left[\frac{(k+1)(2k+5)}{(k+2)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2}\right] - \frac{1}{2}\left[\frac{(2k+5)}{(k+2)(k+3)}\right] \\
&= \frac{1}{2}\left[\frac{3}{2} - \frac{(2[k+1]+3)}{([k+1]+1)\,([k+1]+2)}\right]
\end{aligned}
$$

That is, $\wp(k+1)$ is also true.

[QED]

Note: Additional examples will be given in class.