Limits

Niamh O'Sullivan

February 10, 2009

Outline

Contents

1 Introduction

Introduction

Calculus, to a great extent, is the study of the rate at which quantities change. We would like to answer questions like:

how fast is a population growing?

how fast is a disease spreading?

how fast is a car traveling?

how fast will a ball travel if dropped from a height?

Example

A rock breaks free from the top of the cliff. What is the average speed during the first two secs of the fall?

Experiments have shown that the rock will fall

$$f(t) = 16t^2$$

feet in the first t secs.

The average speed in the first two secs is the distance traveled divided by 2 (length of time interval):

$$= \frac{64}{2} = 32ft/sec.$$

Instantaneous Speed

What is the instantaneous speed at time $t = 2$ (secs)?

We can estimate it by working out the average speed over a shorter time period (including $t = 2$):

from $t = 1$ to $t = 2$ is

$$\frac{f(2) - f(1)}{2 - 1} = \frac{16(2^2) - 16(1^2)}{1} = 48 ft/sec;$$

from $t = 1.9$ to $t = 2$ is $\dfrac{f(2) - f(1.9)}{2 - 1.9} = 62.4 ft/sec;$

from $t = 1.99$ to $t = 2$ is $\dfrac{f(2) - f(1.99)}{2 - 1.99} = 63.84 ft/sec.$

In fact the average speed is tending towards the instantaneous speed of $64 ft/sec$ at $t = 2$ This example encapsulates what we will do in calculus this semester:

The *limit* of the estimates is $64 ft/sec$.

2 Limits

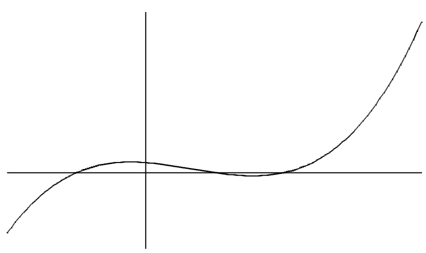

Figure 1: Graph of $f(x) = x^3 - 2x^2 - x + 2$.

In Semester 1 you saw that:

when x got very large , then $f(x)$ also got very large;

or when $x \to \infty$, then $f(x) \to \infty$;

and when $x \to -\infty$, then $f(x) \to -\infty$.

If we pick any large number, say $100,000$ we can find a positive value e.g. $x_0 = 100$ such that $f(100) > 100,000$:

$$f(100) = 1,000,000 - 20,000 - 100 + 2 = 979,902.$$

Likewise if we pick any negative number, say $-100,000$ we can find a negative value e.g. $x_1 = -100$ such that $f(-100) < -100,000$:

$$f(-100) = -1,000,000 - 20,000 + 100 + 2 = -1,019,902.$$

A more formal way of saying this is that $f(x)$ tends to infinity as x tends to infinity, and we write

$$\lim_{x \to \infty} f(x) = \infty.$$

Also $f(x)$ tends to minus infinity as x tends to minus infinity, and we write

$$\lim_{x \to -\infty} f(x) = -\infty.$$

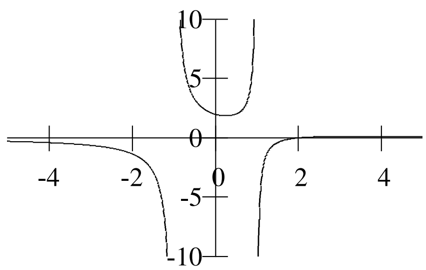

Figure 2: Graph of $h(x) = \frac{x-2}{x^2-1}$.

This function is not defined when $x^2 - 1 = 0$ i.e. when $x = \pm 1$. In order to draw this graph we need to know the behavior of $h(x)$ close to these values and also the behavior as $x \to \pm\infty$.

At the end of this section you should be able to decided whether

(i) a function tends to plus or minus infinity, or to a real limit as x tends to infinity;

(ii) a function tends to plus or minus infinity, or to a real limit as x tends to minus infinity;

(iii) a function tends to plus or minus infinity, or to a real limit as x tends to a given real number.

We will be looking at polynomials and rational functions.

Limit as x tends to infinity

Example 1.1

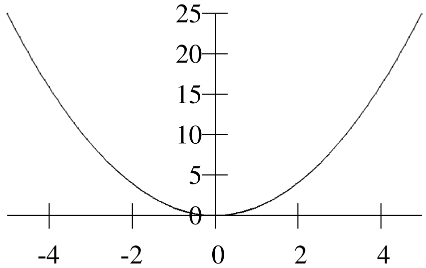

Figure 3: Graph of $f(x) = x^2$.

(i) As x gets very large, $f(x) = x^2$ does not closer to any particular number- it just gets larger and larger. At some point $f(x)$ will be larger than any number we choose and stay larger. In this case we say that $f(x)$ tends to infinity as x tends to infinity and write

$$\lim_{x\to\infty} x^2 = \infty \quad \text{or} \quad f(x) \to \infty, \quad \text{as} \quad x \to \infty.$$

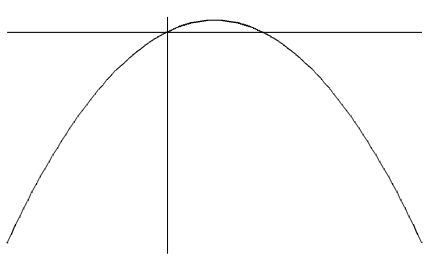

Figure 4: Graph of $g(x) = -x^2 + 3x$.

(ii) As x gets very large, $g(x) = -x^2+3x$ does not closer to any particular number-it just gets more negative.

At some point $g(x)$ will be more negative than any number we choose and stay more negative.

In this case we say that $g(x)$ tends to minus infinity as x tends to infinity and write

$$\lim_{x\to\infty} g(x) = -\infty \text{ or } g(x) \to -\infty, \text{ as } x \to \infty.$$

Note

$$\lim_{x\to\infty} (-x^2) = -\infty$$

and

$$\lim_{x\to\infty} 3x = \infty.$$

However $-x^2 \to -\infty$ at a quicker rate than $3x \to \infty$, for example

$$-(1000)^2 = -1,000,000, \quad 3 \times 1,000 = 3,000, \quad g(1,000) = -997,000.$$

So that the behavior of $-x^2$ (as $x \to \infty$) determines the behavior of $g(x)$ (as $x \to \infty$)

Limits

The function $f(x)$ tends to infinity as x tends to infinity if, however large a number we chose, $f(x)$ gets larger and stays larger than this number, no matter how large x becomes. We write

$$\lim_{x\to\infty} f(x) = \infty.$$

The function $f(x)$ tends to minus infinity as x tends to infinity if, however large and negative a number we chose, $f(x)$ gets more negative and stays more negative than this number, no matter how large x becomes. We write

$$\lim_{x\to\infty} f(x) = -\infty.$$

Definition 1.2

Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, be a polynomial. If $n = 0$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} a_0 = a_0.$$

If $n > 0$, then

$$\lim_{x\to\infty} f(x) = \begin{cases} \infty & \text{if } a_n > 0; \\ -\infty & \text{if } a_n < 0. \end{cases}$$

$$\lim_{x\to\infty} (-2x^4 + 1,000x^3 + 5x + 234) =$$

$$\lim_{x\to\infty} (x^{16} - 2,345,000x^{10} + 567x^5 + 4x) =$$

$$\lim_{x\to\infty} (-8x^{99} + 563x^{98}) =$$

Rational functions

Example 1.3

Let $y = f(x) = \dfrac{1}{x}$.

What is $\lim_{x\to\infty} y = \lim_{x\to\infty} f(x)$?

As x gets larger, $f(x)$ gets smaller:

x	$f(x)$
100	0.01
$1,000$	0.001
$1,000,000$	0.000001

In fact as x gets larger, $f(x)$ gets closer to zero: no matter how small a distance we choose $f(x)$ gets closer than this distance to 0 and stays closer, no matter how large x becomes.

We say that $f(x)$ tends to 0 as x tends to infinity and write

$$\lim_{x\to\infty} f(x) = 0 \text{ or } f(x) \to 0, \text{ as } x \to \infty.$$

1.4 Real Limits

In general the function $f(x)$ has a real limit l as x tends to infinity if, however small a distance we choose, $f(x)$ gets closer than this distance to l and stays closer, no matter how large x becomes and we write

$$\lim_{x\to\infty} f(x) = l \text{ or } f(x) \to l, \text{ as } x \to \infty.$$

Note for any natural number n

$$\lim_{x\to\infty} \frac{1}{x^n} = 0.$$

Example 1.5

(i) Let $f(x) = x - 2$ and $g(x) = x^2 - 1$.

So

$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x).$$

Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x-2}{x^2-1}.$$

What is

$$\lim_{x\to\infty} h(x)?$$

x	$f(x)$	$g(x)$	$h(x)$
10	8	99	0.08
100	98	9,999	0.0098
1,000	998	999,999	0.000998

So as

$$x \to \infty, \quad h(x) \to 0.$$

The reason is $g(x) \to \infty$ at a much faster rate than $f(x) \to \infty$ and hence determines the behavior of $h(x)$ (as $x \to \infty$.)

Again the limit is determined by the highest power of x and where it occurs.

As we are assuming x is large (and hence non-zero) we can divide through by x^2 (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x-2}{x^2-1} = \frac{x/x^2 - 2/x^2}{x^2/x^2 - 1/x^2} = \frac{1/x - 2/x^2}{1 - 1/x^2}.$$

As $x \to \infty$, $1/x \to 0$, $\;1/x^2 \to 0$ and $1 \to 1$.

Therefore
$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1/x - 2/x^2}{1 - 1/x^2} = \frac{0}{1} = 0.$$

(ii) Let $f(x) = x^2 - 5$ and $g(x) = 4x^2 - 1$.

So
$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x).$$

Set
$$h(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 5}{4x^2 - 1}.$$

What is
$$\lim_{x\to\infty} h(x)?$$

x	$f(x)$	$g(x)$	$h(x)$
10	95	399	0.2381
100	$9,995$	$39,999$	0.2499
$1,000$	$999,995$	$3,999,999$	0.25

So as
$$x \to \infty, \quad h(x) \to \frac{1}{4}.$$

The reason is $g(x) \to \infty$ at about 4 times the rate as $f(x) \to \infty$.

As we are assuming x is large (and hence non-zero) we can divide through by x^2 (the highest power of x occurring in the denominator) to get:
$$h(x) = \frac{x^2 - 5}{4x^2 - 1} = \frac{x^2/x^2 - 5/x^2}{4x^2/x^2 - 1/x^2} = \frac{1 - 5/x^2}{4 - 1/x^2}.$$

As $x \to \infty$, $1/x^2 \to 0$, $1 \to 1$ and $4 \to 4$.

Therefore
$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{1 - 5/x^2}{4 - 1/x^2} = \frac{1}{4}.$$

(iii) Let $f(x) = x^3 + x$ and $g(x) = 10x + 3$.

So
$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x).$$

Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x^3 + x}{10x + 3}.$$

What is

$$\lim_{x \to \infty} h(x)?$$

x	$f(x)$	$g(x)$	$h(x)$
10	$1,010$	103	9.8
100	$1,000,100$	1003	997.1
$1,000$	$1,000,001,000$	$10,003$	999790.1

So as

$$x \to \infty, \quad h(x) \to \infty.$$

The reason is $f(x) \to \infty$ at a much faster rate than $g(x) \to \infty$.

As we are assuming x is large (and hence non-zero) we can divide through by x (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x^3 + x}{10x + 3} = \frac{x^3/x + x/x}{10x/x + 3/x} = \frac{x^2 + 1}{10 + 3/x}.$$

As $x \to \infty$, $x^2 \to \infty$, $1 \to 1$, $10 \to 10$ and $3/x \to 0$.

Therefore

$$\lim_{x \to \infty} h(x) = \lim_{x \to \infty} \frac{x^2 + 1}{10 + 3/x} = \infty.$$

Limits

The function $f(x)$ tends to infinity as x tends to minus infinity if, however large a number we chose, $f(x)$ gets larger and stays larger than this number, no matter how large and negative x becomes. We write

$$\lim_{x \to -\infty} f(x) = \infty.$$

The function $f(x)$ tends to minus infinity as x tends to minus infinity if, however large and negative a number we chose, $f(x)$ gets more negative and stays more negative than this number, no matter how large and negative x becomes. We write

$$\lim_{x \to -\infty} f(x) = -\infty.$$

1.6 Limits of Rational functions

Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0,$$

$$g(x) = b_m x^m + \cdots + b_1 x + b_0$$

where $a_n,\ b_m \neq 0$, be polynomials. Set

$$h(x) = \frac{f(x)}{g(x)}.$$

Then

$$\lim_{x\to\infty} h(x) = \begin{cases} \lim_{x\to\infty} f(x) = \pm\infty & \text{if } n > m \\ \dfrac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

1.7 Examples

$$\lim_{x\to\infty} \frac{-x^5 + 4x^3 - 100}{99x^7 + 1} =$$

$$\lim_{x\to\infty} \frac{-x^5 + 4x^3 - 100}{46x^3 - 5x^2 + 198x - 345} =$$

$$\lim_{x\to\infty} \frac{5x^{33} + 46x^{31} - 100x^{22}}{941x^{33} + 4} =$$

$$\lim_{x\to\infty} \frac{-x^6 + 2x^4 + 10x}{3x^6 + x^2} =$$

$$\lim_{x\to\infty} \frac{x^{56} + 34x^{29} - 23x^4 + 1}{x^{27} + 5x^4 - 3x^2 + 102} =$$

$$\lim_{x\to\infty} \frac{1,000,000x^4 + 7x^2 + 50}{-8x^7 + 1} =$$