Limit as x tends to minus infinity

Example 1.8

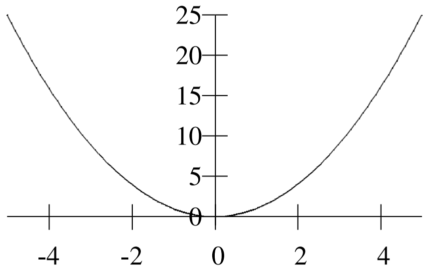

Figure 1: Graph of $f(x) = x^2$.

(i) As x gets more and more negative, $f(x) = x^2$ does not closer to any particular number- it just gets larger. At some point $f(x)$ will be larger than any number we choose and stay larger. In this case we say that $f(x)$ tends to infinity as x tends to minus infinity and write

$$\lim_{x \to -\infty} x^2 = \infty \text{ or } f(x) \to \infty, \text{ as } x \to -\infty.$$

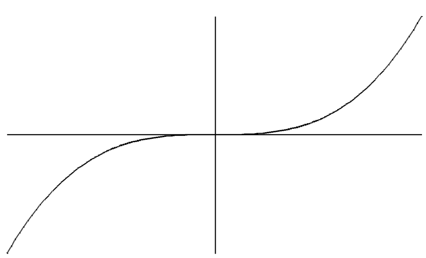

Figure 2: Graph of $g(x) = x^3$.

(ii) As x gets more negative, $g(x) = x^3$ does not closer to any particular number-it just gets more negative.

At some point $g(x)$ will be more negative than any number we choose and stay more negative.

In this case we say that $g(x)$ tends to minus infinity as x tends to minus infinity and write

$$\lim_{x \to -\infty} g(x) = -\infty \text{ or } g(x) \to -\infty, \text{ as } x \to -\infty.$$

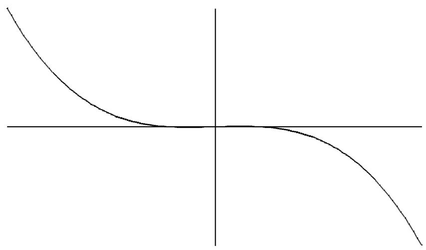

Figure 3: Graph of $h(x) = -x^3 + 5x$.

Note $\lim_{x \to -\infty} (-x^3) = \infty$ and $\lim_{x \to -\infty} 5x = -\infty$.

However $-x^3 \to \infty$ at a much quicker rate than $5x \to -\infty$.

x	$-x^3$	$5x$	$h(x)$
-10	$1,000$	-50	950
-100	$1,000,000$	-500	$999,500$
$-1,000$	$1,000,000,000$	$-5,000$	999995000

So that the behavior of $-x^3$ (as $x \to \infty$) determines the behavior of $h(x)$ (as $x \to \infty$)

Limits

The function $f(x)$ tends to infinity as x tends to minus infinity if, however large a number we chose, $f(x)$ gets larger and stays larger than this number, no matter how large and negative x becomes. We write

$$\lim_{x \to -\infty} f(x) = \infty.$$

The function $f(x)$ tends to minus infinity as x tends to infinity if, however large and negative a number we chose, $f(x)$ gets more negative and stays more negative than this number, no matter how large and negative x becomes. We write

$$\lim_{x \to -\infty} f(x) = -\infty.$$

1.9

Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, is a polynomial. If $n = 0$, then

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} a_0 = a_0.$$

If $n > 0$, then

$$\lim_{x \to -\infty} f(x) = \begin{cases} \infty & \text{if } a_n > 0 \text{ and } n \text{ is even;} \\ -\infty & \text{if } a_n < 0 \text{ and } n \text{ is even;} \\ -\infty & \text{if } a_n > 0 \text{ and } n \text{ is odd;} \\ \infty & \text{if } a_n < 0 \text{ and } n \text{ is odd.} \end{cases}$$

$$\lim_{x \to -\infty} (-2x^4 + 1{,}000x^3 + 5x + 234) = -\infty.$$

$$\lim_{x \to -\infty} (x^{16} - 2{,}345{,}000x^{10} + 567x^5 + 4x) = \infty.$$

$$\lim_{x \to -\infty} (-8x^{99} + 563x^{98}) = \infty.$$

$$\lim_{x \to -\infty} (77x^{21} + 34x^9) = -\infty.$$

Rational functions

Let $y = f(x) = \dfrac{1}{x}$.

What is $\lim_{x \to -\infty} y = \lim_{x \to -\infty} f(x)$?

As x gets more negative, $f(x)$ gets smaller:

x	$f(x)$
-100	-0.01
$-1{,}000$	-0.001
$-1{,}000{,}000$	-0.000001

In fact as x gets more negative, $f(x)$ gets closer to zero: no matter how small a distance we choose $f(x)$ gets closer than this distance to 0 and stays closer, no matter how large and negative x becomes.

We say that $f(x)$ tends to 0 as x tends to minus infinity and write

$$\lim_{x \to -\infty} f(x) = 0 \quad \text{or} \quad f(x) \to 0, \quad \text{as} \quad x \to -\infty.$$

1.10 Limits

In general the function $f(x)$ has a real limit l as x tends to minus infinity if, however small a distance we choose, $f(x)$ gets closer than this distance to l and stays closer, no matter how large and negative x becomes and we write

$$\lim_{x \to -\infty} f(x) = l \ \text{ or } \ f(x) \to l, \ \text{ as } \ x \to -\infty.$$

Note for any natural number n

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0.$$

Example 1.11

(i) Let $f(x) = x - 2$ and $g(x) = x^2 - 1$.

So

$$\lim_{x \to -\infty} f(x) = -\infty, \ \lim_{x \to -\infty} g(x) = \infty.$$

Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x-2}{x^2-1}.$$

What is

$$\lim_{x \to -\infty} h(x)?$$

x	$f(x)$	$g(x)$	$h(x)$
-10	-12	99	-0.1212
-100	-102	$9,999$	-0.0102
$-1,000$	$-1,002$	$999,999$	-0.001

So as

$$x \to -\infty, \ \ h(x) \to 0.$$

The reason is $g(x) \to \infty$ at a much faster rate than $f(x) \to -\infty$ and hence determines the behavior of $h(x)$ (as $x \to -\infty$.)

Again the limit is determined by the highest power of x and where it occurs.

As we are assuming x is large and negative (and hence non-zero) we can divide through by x^2 (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x-2}{x^2-1} = \frac{x/x^2 - 2/x^2}{x^2/x^2 - 1/x^2} = \frac{1/x - 2/x^2}{1 - 1/x^2}.$$

As $x \to -\infty$, $1/x \to 0$, $\ 1/x^2 \to 0$ and $1 \to 1$.

Therefore

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{1/x - 2/x^2}{1 - 1/x^2} = \frac{0}{1} = 0.$$

(ii) Let $f(x) = x^2 - 5$ and $g(x) = 4x^2 - 1$.

So

$$\lim_{x \to -\infty} f(x) = \infty = \lim_{x \to -\infty} g(x).$$

Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 5}{4x^2 - 1}.$$

What is

$$\lim_{x \to -\infty} h(x)?$$

As we are assuming x is large (and hence non-zero) we can divide through by x^2 (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x^2 - 5}{4x^2 - 1} = \frac{x^2/x^2 - 5/x^2}{4x^2/x^2 - 1/x^2} = \frac{1 - 5/x^2}{4 - 1/x^2}.$$

As $x \to -\infty$, $1/x^2 \to 0$, $1 \to 1$ and $4 \to 4$.

Therefore

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{1 - 5/x^2}{4 - 1/x^2} = \frac{1}{4}.$$

(iii) Let $f(x) = x^3 + x$ and $g(x) = 10x + 3$.

So

$$\lim_{x \to -\infty} f(x) = -\infty = \lim_{x \to -\infty} g(x).$$

Set

$$h(x) = \frac{f(x)}{g(x)} = \frac{x^3 + x}{10x + 3}.$$

What is

$$\lim_{x \to -\infty} h(x)?$$

As we are assuming x is large (and hence non-zero) we can divide through by x (the highest power of x occurring in the denominator) to get:

$$h(x) = \frac{x^3 + x}{10x + 3} = \frac{x^3/x + x/x}{10x/x + 3/x} = \frac{x^2 + 1}{10 + 3/x}.$$

As $x \to -\infty$, $x^2 \to \infty$, $1 \to 1$, $10 \to 10$ and $3/x \to 0$.

Therefore

$$\lim_{x\to-\infty} h(x) = \lim_{x\to\infty} \frac{x^2+1}{10+3/x} = \infty.$$

Method

$$\begin{aligned}\lim_{x\to-\infty} h(x) &= \lim_{x\to-\infty} \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} \\ &= \lim_{x\to-\infty} \frac{a_n x^n/x^m + \cdots + a_0/x^m}{b_m + \cdots + b_0/x^m}.\end{aligned}$$

We divided both the numerator and denominator by x^m (the highest power of x in the denominator). For any natural number k

$$\lim_{x\to-\infty} 1/x^k = 0.$$

For $n = m$,

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{a_n + \cdots + a_0/x^m}{b_m + \cdots + b_0/x^m} = \frac{a_n}{b_m}.$$

For $n < m$,

$$\lim_{x\to-\infty} h(x) = \lim_{x\to-\infty} \frac{a_n/x^{m-n} + \cdots + a_0/x^m}{b_m + \cdots + b_0/x^m} = 0.$$

For $n > m$,

$$\begin{aligned}\lim_{x\to-\infty} h(x) &= \lim_{x\to-\infty} \frac{a_n x^{n-m} + \cdots + a_0/x^m}{b_m + \cdots + b_0/x^m} \\ &= \lim_{x\to-\infty} \frac{a_n}{b_m} x^{n-m} = \pm\infty.\end{aligned}$$

1.12 Limits of Rational functions

Let

$$\begin{aligned}f(x) &= a_n x^n + \cdots + a_1 x + a_0, \\ g(x) &= b_m x^m + \cdots + b_1 x + b_0\end{aligned}$$

where a_n, $b_m \neq 0$, be polynomials. Set

$$h(x) = \frac{f(x)}{g(x)}.$$

Then

$$\lim_{x\to-\infty} h(x) = \begin{cases} \lim\limits_{x\to-\infty} \dfrac{a_n}{b_m} x^{n-m} = \pm\infty & \text{if } n > m \\ \dfrac{a_n}{b_m} & \text{if } n = m \\ 0 & \text{if } n < m \end{cases}$$

1.13 Examples

$$\lim_{x\to-\infty} \frac{-x^5 + 4x^3 - 100}{99x^7 + 1} = 0$$

$$\lim_{x\to-\infty} \frac{-x^5 + 4x^3 - 100}{46x^3 - 5x^2 + 198x - 345} = -\infty$$

$$\lim_{x\to-\infty} \frac{5x^{33} + 46x^{31} - 100x^{22}}{941x^{33} + 4} = \tfrac{5}{941}$$

$$\lim_{x\to-\infty} \frac{-x^6 + 2x^4 + 10x}{3x^6 + x^2} = -\tfrac{1}{3}$$

$$\lim_{x\to-\infty} \frac{x^{56} + 34x^{29} - 23x^4 + 1}{x^{27} + 5x^4 - 3x^2 + 102} = -\infty$$

$$\lim_{x\to-\infty} \frac{1,000,000x^4 + 7x^2 + 50}{-8x^7 + 1} = 0$$

1.14 Rules for Limits as $x \to \pm\infty$

If L, M and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L, \quad \lim_{x\to\pm\infty} g(x) = M,$$

then

(i) Sum Rule: $\lim\limits_{x\to\pm\infty} f(x) + g(x) = L + M$.

(ii) Difference Rule: $\lim\limits_{x\to\pm\infty} f(x) - g(x) = L - M$.

(iii) Product Rule: $\lim\limits_{x\to\pm\infty} f(x)g(x) = LM$.

(iv) Constant Multiple Rule: $\lim\limits_{x\to\pm\infty} kf(x) = kL$.

(v) Quotient Rule: If $M \neq 0$, then $\lim\limits_{x\to\pm\infty} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.