Limits

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Outline

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Limits as x tends to a real number

Examples 1.15 (i)

Let $f(x) = x + 1$.

As x gets closer to 1, $f(x)$ gets closer to $2 = f(1)$.

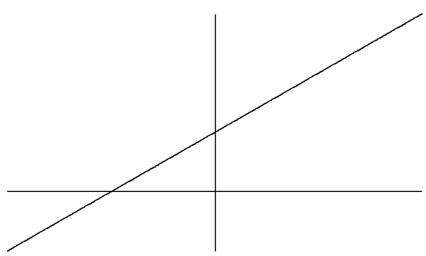

Figure 1: Graph of $f(x) = x + 1$.

(ii)

Let $g(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x-1)(x+1)}{x-1}$.

Then g is not defined when $x = 1$.

However for $x \neq 1$, $g(x) = x + 1$ and as x gets closer to 1, $g(x)$ gets closer to 2.

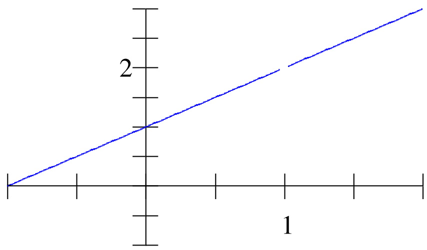

Figure 2: Graph of $g(x) = \dfrac{x^2 - 1}{x - 1}$.

Definition 1.16

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself.* We say that f approaches the limit l (l a real number)as x approaches x_0 if however small a distance we choose, $f(x)$ gets closer than this distance to l for x sufficiently close to (but not equal to) x_0. We write

$$\lim_{x \to x_0} f(x) = l$$

or

$$f(x) \to l \ \text{ as } \ x \to x_0.$$

Example 1.17

(i) Let $f(x) = 5$, for all x. What is $\lim_{x \to 2} f(x)$?

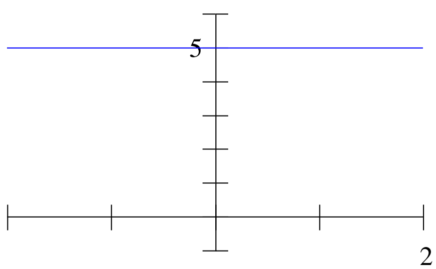

Figure 3: Graph of $f(x) = 5$.

$\lim_{x \to 2} f(x) = 5$.
Fact: If $f(x) = k$, for some constant k, then $\lim_{x \to x_0} f(x) = k$, for any x_0.
(ii) What is $\lim_{x \to 3} x^2$?
$\lim_{x \to 3} x^2 = 9$.

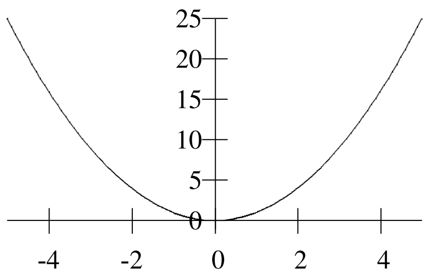

Figure 4: Graph of $y = x^2$.

Fact: If f is any polynomial, then $\lim_{x \to x_0} f(x) = f(x_0)$, for any x_0.

(iii) Let

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & \text{if } x \neq 1; \\ 1 & \text{if } x = 1. \end{cases}$$

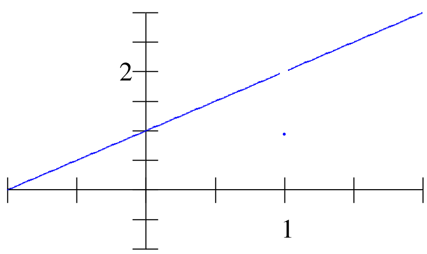

Figure 5: Graph of $f(x)$.

1.18 Rules

If $L, M,$ c and k are real numbers and

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = M,$$

then

(i) Sum Rule: $\lim_{x \to c} f(x) + g(x) = L + M$.

(ii) Difference Rule: $\lim_{x \to c} f(x) - g(x) = L - M$.

(iii) Product Rule: $\lim_{x \to c} f(x)g(x) = LM$.

(iv) Constant Multiple Rule: $\lim_{x \to c} kf(x) = kL$.

(v) Quotient Rule: If $M \neq 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

If f is a polynomial, then $\lim_{x \to c} f(x) = f(a) = L$.

If f and g are polynomials and $g(c) \neq 0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0.$$

Example 1.19 Eliminating zero denominators algebraically

(i) Find $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$.

We cannot substitute $x = 1$ as it makes the denominator zero.

We test the numerator to see if it is zero at $x = 1$ (if so it has a factor of $(x - 1)$ in common with the denominator):

$$1^2 + 1 - 2 = 0.$$

We can cancel the $(x - 1)$'s to get:

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} &= \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)x} \\ &= \lim_{x \to 1} \frac{x+2}{x} = 3. \end{aligned}$$

(ii) Find $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 - 4}$.

We cannot substitute $x = -2$ as it makes the denominator zero.

We test the numerator to see if it is zero at $x = -2$ (if so it has a factor of $(x+2)$ in common with the denominator):

$$(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0.$$

We can cancel the $(x+2)$'s to get:

$$\begin{aligned}\lim_{x\to -2} \frac{x^2+5x+6}{x^2-4} &= \lim_{x\to -2} \frac{(x+3)(x+2)}{(x-2)(x+2)} \\ &= \lim_{x\to -2} \frac{x+3}{x-2} = -\frac{1}{4}.\end{aligned}$$

Examples 1.20

(i) Let $f(x) = \dfrac{1}{x}$.

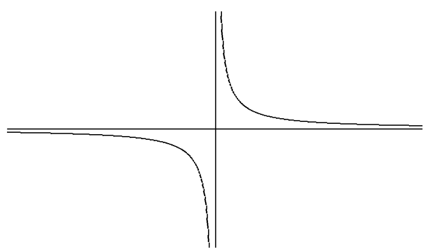

Figure 6: Graph of $f(x) = \dfrac{1}{x}$.

As we can see from the graph $f(x) < 0$, if $x < 0$ and $f(x) > 0$, if $x > 0$.

x	$f(x)$
-0.1	-10
-0.01	-100
-0.001	$-1,000$
-0.0001	$-10,000$
0.1	10
0.01	100
0.001	$1,000$
0.0001	$10,000$

If x is close to but *less than* 0, $f(x) \to -\infty$, as $x \to 0$.

If x is close to but *greater than* 0, $f(x) \to \infty$, as $x \to 0$.

In this case $\dfrac{1}{x} \to -\infty$ as x approaches 0 from the left (x is less than 0), we write

$$\lim_{x\to 0^-} \frac{1}{x} = \lim_{x\uparrow 0} \frac{1}{x} = -\infty.$$

Likewise $\frac{1}{x} \to \infty$ as x approaches 0 from the right (x is greater than 0), we write

$$\lim_{x \to 0^+} \frac{1}{x} = \lim_{x \downarrow 0} \frac{1}{x} = \infty.$$

The line $x = 0$ is called a vertical asymptote of the graph.

The line $y = 0$ is called a horizontal asymptote of the graph.

(ii) Let $f(x) = \frac{x+3}{x+2}$. Then f is not defined when $x = -2$.

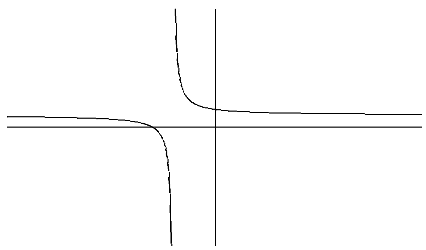

Figure 7: Graph of $f(x) = \frac{x+3}{x+2}$.

Definition 1.21

A line $y = b$ is called a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \ \text{ or } \ \lim_{x\to-\infty} f(x) = b.$$

A line $x = a$ is called a vertical asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\uparrow a} f(x) = \pm\infty \ \text{ or } \ \lim_{x\downarrow a} f(x) = \pm\infty.$$

Let

$$h(x) = \frac{f(x)}{g(x)}$$

be a rational function. If $g(a) = 0$, but $f(a) \neq 0$, then the line $x = a$ is a vertical asymptote of the graph of h and

$$\lim_{x\uparrow a} h(x) = \pm\infty, \ \lim_{x\downarrow a} h(x) = \pm\infty.$$

Furthermore if $k(x) = f(x)g(x)$, then

$$h(x) > 0 \Longleftrightarrow k(x) > 0,$$

$$h(x) < 0 \Longleftrightarrow k(x) < 0.$$

To determine $\lim_{x\uparrow a} h(x)$, we look at $k(x)$ for x close to but less than a.

To determine whether $\lim_{x\downarrow a} h(x)$, we look at $k(x)$ for x close to but greater than a.

Example 1.22

(i) Let $f(x) = -\dfrac{8}{x^2 - 4}$. Then f is not defined when $x = \pm 2$.

$f(x) = -\dfrac{8}{x^2 - 4}$ is not defined, when $x^2 - 4 = 0$ or $x = \pm 2$.

If x is close to but *less than* -2, $x^2 - 4 > 0$ and so $f(x) = -8(x^2 - 4) < 0$ and $f(x) \to -\infty$, as $x \uparrow -2$:

$$\lim_{x\uparrow -2} f(x) = -\infty.$$

If x is close to but *greater than* 2, $x^2 - 4 > 0$ and so $f(x) = -8(x^2 - 4) < 0$ and $f(x) \to -\infty$, as $x \downarrow 2$:

$$\lim_{x\downarrow 2} f(x) = -\infty.$$

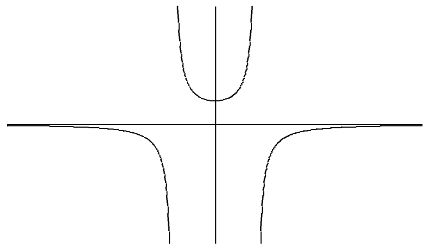

Figure 8: Graph of $f(x) = -\dfrac{8}{x^2 - 4}$.

If $-2 < x < 2$, then $x^2 - 4 > 0$ so $f(x) > 0$ and

$$\lim_{x \downarrow -2} f(x) = \infty = \lim_{x \uparrow 2} f(x).$$

(ii) Let $f(x) = \dfrac{11x + 2}{2x^3 - 1}$. f is not defined when $x = \frac{1}{\sqrt[3]{2}}$.

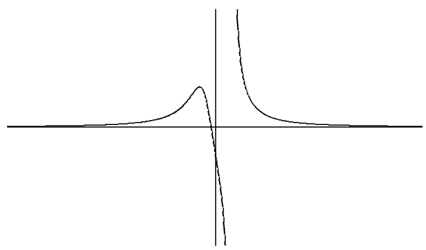

Figure 9: Graph of $f(x) = \dfrac{11x + 2}{2x^3 - 1}$.