The natural logarithm and exponential functions

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4.1 Natural Logarithm

If $n \neq -1$, then $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$.

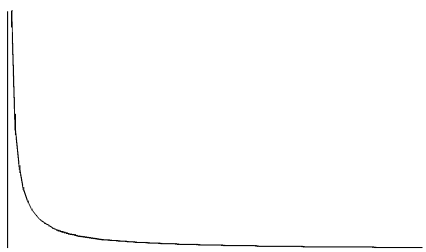

Figure 1: Graph of $f(x) = x^{-1}$, for $x > 0$.

For $x > 1$, the integral $\int_1^x \frac{1}{t} \, dt$ represents the area under the curve $y = \frac{1}{t}$ and above the t-axis, between $t = 1$ and $t = x$.

4.1 Natural Logarithm

For any $x > 0$, set

$$\ln(x) = \int_1^x \frac{1}{t} \, dt.$$

The function ln is called the natural logarithm. By 3.16, (F.T.O.C. Part I)

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x},$$

for $x > 0$. So the natural logarithm is an antiderivative of x^{-1} on $(0, \infty)$.

1.1 Properties

4.2 Properties

1. $\ln(1) = \int_1^1 \frac{1}{t}\, dt = 0.$

2. If $x > 1$, then $\ln(x) > 0$. This is apparent from the area interpretation but also from 3.15 (Property 7).

3. If $0 < x < 1$, then

$$\ln(x) = \int_1^x \frac{1}{t}\, dt = -\int_x^1 \frac{1}{t}\, dt < 0,$$

 3.15 (Properties 1 and 7).

4. $\int \frac{1}{x}\, dx = \ln(|x|) + C$, for $x \neq 0$.

 When $x > 0$, then $|x| = x$ and

$$\frac{d}{dx}(\ln(|x|)) = \frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

 When $x < 0$, then $|x| = -x$ and we set $u = -x > 0$. B y the Chain rule

$$\frac{d}{dx}(\ln(|x|)) = \frac{d}{dx}(\ln(u)) = \frac{d}{du}(\ln(u)\frac{du}{dx} = -\frac{1}{u} = \frac{1}{x}.$$

5. If u is a differentiable function that is never zero, then

$$\int \frac{1}{u}\, du = \ln(|u|) + C.$$

 If f is a differentiable function which maintains a constant sign on the domain given for it, then

$$\int \frac{f'(x)}{f(x)}\, dx = \ln(|f(x)|) + C.$$

6. For any u, $v > 0$,

$$\ln(uv) = \ln(u) + \ln(v).$$

$\ln(v) = \int_1^v \frac{1}{t}\, dt.$

Make the substitution $s = ut$ so $ds = u\, dt$, when $t = 1,\ s = u$, when $t = v,\ s = uv$ and

$$\ln(v) = \int_1^v \frac{1}{t}\, dt = \int_1^v \frac{u}{ut}\, dt = \int_u^{uv} \frac{1}{s}\, ds = \int_u^{uv} \frac{1}{t}\, dt.$$

Therefore, by 3.15 (Property 5),

$$\ln(u) + \ln(v) = \int_1^u \frac{1}{t}\, dt + \int_u^{uv} \frac{1}{t}\, dt = \int_1^{uv} \frac{1}{t}\, dt = \ln(uv).$$

7. For any $u,\ v > 0,$

$$\ln\left(\frac{u}{v}\right) = \ln(u) - \ln(v).$$

From 6,

$$\ln(u) = \ln\left(\frac{u}{v}v\right) = \ln\left(\frac{u}{v}\right) + \ln(v).$$

8. For any $u > 0$, $\ln\left(\frac{1}{u}\right) = -\ln(u).$

From 7,

$$\ln\left(\frac{1}{u}\right) = \ln(1) - \ln(u) = -\ln(u).$$

9. For any rational number n,

$$\ln(x^n) = n\ln(x).$$

If $n = 0$, then $x^0 = 1$ and $\ln(x^0) = \ln(1) = 0 = 0.\ln(x).$

If $n \neq 0$, set $u = x^n,$ then, by the Chain Rule,

$$\begin{aligned}\frac{d}{dx}(\ln(x^n)) &= \frac{d}{dx}(\ln(u)) = \frac{du}{dx}\frac{d(ln(u))}{du} \\ &= \frac{nx^{n-1}}{u} = \frac{nx^{n-1}}{x^n} = \frac{n}{x} \\ &= \frac{d}{dx}(n\ln(x)).\end{aligned}$$

By 3.3, $\ln(x^n) = n\ln(x) + c$, for some constant c. Substituting $x = 1$, we see that $c = 0$ and the property is true.

10. $\ln$ is an increasing function.

    This follows as
$$\frac{d(\ln(x))}{dx} = \frac{1}{x} > 0,$$
    for $x > 0$.

11. If $u, v > 0$ and $\ln(u) = \ln(v)$, then $u = v$.

    This follows directly from 10.

12. $\lim_{x\to\infty} \ln(x) = \infty$.

    As $\ln$ is increasing, we only need to show that given any integer k we can find x with $\ln(x) > k$.

    If $x > 2^{2k}$, then
$$\ln(x) > \ln(2^{2k}) = 2k\ln(2).$$

    $\ln(2) = \int_1^2 \frac{1}{t}\,dt \geq \frac{1}{2}(2-1) = \frac{1}{2}$, by the Max-Min inequality (3.15 Property 5).

    So $\ln(x) > k$ and $\lim_{x\to\infty} \ln(x) = \infty$.

13. $\lim_{x\to 0+} \ln(x) = -\infty$.
$$\lim_{x\to 0+} \ln(x) = \lim_{u\to\infty} \ln\left(\frac{1}{u}\right) = \lim_{u\to\infty} (-\ln(u)) = -\infty.$$

1.2 Derivatives

4.3 Derivatives involving $\ln$

Find the derivative of the following functions:

(i) $f(x) = \ln^n(x) = (\ln(x))^n$, for any non-zero rational number n.

We set $u = \ln(x)$, then $f = u^n$ and by the Chain Rule
$$\begin{aligned}\frac{df}{dx} &= \frac{du^n}{du}.\frac{du}{dx}\\ &= nu^{n-1}.\frac{1}{x}\\ &= \frac{n\ln^{n-1}(x)}{x}.\end{aligned}$$

(ii) $f(x) = \ln(u)$, for some differentiable function u.

Again we use the Chain Rule

$$\begin{aligned}\frac{df}{dx} &= \frac{d(\ln(u))}{du}.\frac{du}{dx} \\ &= \frac{u'}{u}.\end{aligned}$$

(iii)

$$f(x) = \ln^5(x) \Longrightarrow f'(x) = \frac{5\ln^4(x)}{x}.$$

(iv) $f(x) = \ln(x^6)$.

Here $f(x) = 6\ln(x)$ so $f'(x) = \frac{6}{x}$.

(v) $f(x) = \ln(x^3 - 2x)$

Here $u = x^3 - 2x$ so $u' = 3x^2 - 2$ and

$$f'(x) = \frac{3x^2 - 2}{x^3 - 2x}.$$

(vi) $f(x) = x^5 \ln(x)$

Use the product rule with $u = x^5$, $v = \ln(x)$. So $u' = 5x^4$, $v' = \frac{1}{x}$ and

$$f'(x) = \frac{x^5}{x} + 5x^4\ln(x) = x^4(1 + 5\ln(x)).$$

(vii) $f(x) = \ln(x^2\sqrt{x^4 + 5})$

Here $f(x) = \ln(x^2) + \ln((x^4+5)^{1/2}) = 2\ln(x) + \frac{1}{2}\ln(x^4 + 5)$.

So

$$\begin{aligned}f'(x) &= \frac{d}{dx}(2\ln(x)) + \frac{d}{dx}\left(\frac{1}{2}\ln(x^4 + 5)\right) \\ &= \frac{2}{x} + \frac{4x^3}{2x^4 + 10}.\end{aligned}$$

(viii) $f(x) = \ln\left(\frac{1}{x^3}\right)$.

Here $f(x) = \ln\left(x^{-3}\right) = -3\ln(x)$. So

$$f'(x) = -\frac{3}{x}.$$

(ix) $f(x) = \ln\left(\frac{2+x^2}{1+x}\right)$.

Here $f(x) = \ln(2+x^2) - \ln(1+x)$. So

$$f'(x) = \frac{2x}{2+x^2} - \frac{1}{1+x}.$$

1.3 Integrals

4.4 Integrals involving $\ln$

Determine each of the following integrals:

(i) $I = \int \ln(x)\, dx$.

We use integration by parts: set $u = \ln(x)$, $v' = 1$, then $u' = \frac{1}{x}$ and $v = x$.

So

$$\begin{aligned} I &= \int \ln(x).1\, dx = x\ln(x) - \int x\frac{1}{x}dx \\ &= x\ln(x) - \int 1\, dx = x\ln(x) - x + C. \end{aligned}$$

(ii) $I = \int \frac{x}{5-4x^2}\, dx$

Set $f(x) = 5 - 4x^2$, then $f'(x) = -8x$.

So

$$I = -\frac{1}{8}\int \frac{f'(x)}{f(x)}\, dx = -\frac{1}{8}\ln(|f(x)|) + C = -\frac{1}{8}\ln(|5-4x^2|) + C.$$

(iii) $I = \int_4^5 \frac{dt}{\sqrt{t}(4+\sqrt{t})}$.

Set $u(t) = 4 + \sqrt{t}$, then $u'(t) = \frac{1}{2\sqrt{t}}$ and $u(4) = 6$, $u(5) = 4 + \sqrt{5}$.

So

$$\begin{aligned} I &= 2\int_4^5 \tfrac{u'}{u}\,dx = 2\int_6^{4+\sqrt{5}} \tfrac{1}{u}\,du \\ &= 2\left[\ln(|u|)\right]_6^{4+\sqrt{5}} = 2\ln\left(\tfrac{4+\sqrt{5}}{6}\right). \end{aligned}$$

(iv) $\displaystyle\int t^2 \ln(t)\,dt$

Use integration by parts:

$$u = \ln(t),\ u' = \frac{1}{t},\ v' = t^2,\ v = \frac{t^3}{3}.$$

$$\int t^2 \ln(t) = \frac{t^3}{3}\ln(t) - \int \frac{t^2}{3}\,dt = \frac{t^3}{3}\ln(t) - \frac{t^3}{9} + C.$$

(v) $\displaystyle\int \ln\left(\frac{1}{t^2}\right)\,dt$

We use the fact that $\ln\left(\dfrac{1}{t^2}\right) = -2\ln(t)$.

Therefore

$$\int \ln\left(\frac{1}{t^2}\right)\,dt = \int -2\ln(t)\,dt = -\frac{2}{t} + C.$$

2 Exponential functions

4.5 The Exponential function

The natural logarithm function is a $1-1$ function and so we can define an inverse function $\ln^{-1} : \mathbb{R} \to (0,\infty)$ (we will call it $\exp$) as follows

$$exp(x) = y \ \text{ where } \ \ln(y) = x.$$

Let $e = \exp(1)$. Then the following are true:

(i) $\ln(e^n) = n\ln(e) = \ln(\exp(n))$ and so $e^n = \exp(n)$, for any rational number n.

(ii) For any real number x we define $e^x = \exp(x)$.

(iii) $\ln(e^{u+v}) = (u+v) = \ln(e^u) + \ln(e^v) = \ln(e^u e^v)$ and so $e^{u+v} = e^u e^v$.

(iv) $e^{u-v} = \dfrac{e^u}{e^v}$.

(v) $e^{-v} = \dfrac{1}{e^v}$.

4.6 Exponential functions

Let a be any positive real number. Then, for any rational number n

$$a^n = e^{\ln(a^n)} = e^{n\ln(a)}.$$

For any real number x, we define

$$a^x = e^{x\ln(a)}.$$

We can show (see board) that the function $x \mapsto a^x$ (exponential function) satisfies the following properties, for $u, v \in \mathbb{R},\ b > 0$:

$(i)\ \ a^0 = 1.$ $\quad (ii)\ \ a^1 = a.$ $\quad (iii)\ \ a^{u+v} = a^u a^v.$

$(iv)\ \ a^{u-v} = \dfrac{a^u}{a^v}.$ $\quad (v)\ \ a^{-v} = \dfrac{1}{a^v}.$ $\quad (vi)\ \ (ab)^x = a^x b^x.$

$(vii)\ \ \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}.$ $\quad (viii)\ \ (a^u)^v = a^{uv}.$

4.7 Derivative and Integral of e^x

Set $y = e^x$. Then $\ln(y) = x$.

By the Chain Rule,

$$\frac{1}{y}\frac{dy}{dx} = 1.$$

Therefore $\dfrac{dy}{dx} = y$ and

$$\frac{d}{dx}\left(e^x\right) = e^x.$$

If u is a differentiable function of x, then

$$\frac{d}{dx}\left(e^u\right) = \frac{du}{dx}e^u.$$

Hence e^x is an antiderivative of e^x and

$$\int e^x\,dx = e^x + C.$$

4.8 Derivative and Integral of a^x

Set $a^x = e^{x\ln(a)}$.

By the Chain Rule,

$$\begin{aligned}\frac{d}{dx}\left(a^x\right) &= \frac{d}{dx}\left(e^{x\ln(a)}\right) = e^{x\ln(a)}\frac{d}{dx}(x\ln(a)) \\ &= \ln(a)e^{x\ln(a)} = \ln(a)a^x.\end{aligned}$$

As $\dfrac{a^x}{\ln(a)}$ is an antiderivative of a^x and

$$\int a^x\,dx = \frac{a^x}{\ln(a)} + C.$$

4.9 Examples

Find the derivative of the following functions w.r.t. x:

(i) 6^x

$$\frac{d}{dx}\left(6^x\right) = \ln(6)6^x.$$

(ii) 4^x9^x

$$\frac{d}{dx}\left(4^x9^x\right) = \frac{d}{dx}\left(36^x\right) = \ln(36)36^x.$$

(iii) x^2e^{2x}

Use the Product Rule and Chain Rule:

$$\begin{aligned}\frac{d}{dx}\left(x^2e^{2x}\right) &= 2xe^{2x} + x^2\frac{d}{dx}\left(e^{2x}\right) \\ &= 2xe^{2x} + 2x^2e^{2x} = 2xe^{2x}\left(1+x\right).\end{aligned}$$

(iv) e^{x^3-x}.

Use the Chain Rule, set $u = x^3 - x$:

$$\frac{d}{dx}\left(e^{x^3-x}\right) = \frac{du}{dx}e^u = (3x^2-1)e^{x^3-x}.$$

(v) $\displaystyle\int_0^x \left(\frac{e^{t^3} - t^5}{\ln(5t) - t}\right) dt.$

By the F.T.O.C. Part I

$$\frac{d}{dx}\left(\int_0^x \left(\frac{e^{t^3} - t^5}{\ln(5t) - t}\right) dt\right) = \frac{e^{x^3} - x^5}{\ln(5x) - x}.$$

4.10 Integrals involving $\exp$

Determine each of the following integrals:

(i) $\displaystyle\int_0^2 3x^2 e^{x^3}\, dx.$

Here $I = \int u' e^u dx$ where $u = x^3,\ u(0) = 0,\ u(2) = 8.$

Therefore

$$I = \int_0^8 e^u du = [e^u]_0^8 = e^8 - e^0 = e^8 - 1.$$

(ii) $\displaystyle I = \int x^2 e^x\, dx.$

Use integration by parts: let $u = x^2,\ v' = e^x$, then $u' = 2x,\ v = e^x$ and

$$I = \int x^2 e^x\, dx = x^2 e^x - 2\int x e^x\, dx.$$

Need to use integration by parts again to find $\int x e^x\, dx$ (see next tutorial sheet).

(iii) $\displaystyle\int_{-1}^1 7^x\, dx = \left[\tfrac{7^x}{\ln(7)}\right]_{-1}^1 = \tfrac{1}{\ln(7)}\left(7 - \tfrac{1}{7}\right) = \tfrac{48}{7\ln(7)}$, by 4.10.

(iv) $\displaystyle I = \int x^2 7^x\, dx.$

Use integration by parts: let $u = x^2,\ v' = 7^x$, then $u' = 2x,\ v = \dfrac{7^x}{\ln(7)}$ and

$$I = \int x^2 \frac{7^x}{\ln(7)}\, dx = x^2\frac{7^x}{\ln(7)} - \frac{2}{\ln(7)}\int x 7^x\, dx.$$

Use integration by parts again: set $u_1 = x,\ v_1' = 7^x$, then $u_1' = 1,\ v_1 = \dfrac{7^x}{\ln(7)}$ and

$$\int x\frac{7^x}{\ln(7)}\,dx = x\frac{7^x}{\ln(7)} - \frac{1}{\ln(7)}\int 7^x\,dx = \frac{7^x}{\ln(7)}\left(x - \frac{1}{\ln(7)}\right) + C.$$

So

$$\int x^2\frac{7^x}{\ln(7)}\,dx = \frac{7^x}{\ln(7)}\left(x^2 - \frac{2x}{\ln(7)} + \frac{2}{\ln^2(7)}\right) + C.$$

4.11 Exponential Growth and Decay

Suppose that f is a differentiable function of t with $f'(t) = f(t)$.

Then set $F(t) = \dfrac{f(t)}{e^t}$.

By the Quotient Rule

$$F'(t) = \frac{e^t f'(t) - e^t f(t)}{e^{2t}} = 0.$$

Therefore $F(t) = c$, for some constant c and

$$f(t) = ce^t.$$

Suppose that y is a differentiable function of t with $y' = \alpha y$, for some constant α.

Set $u = \alpha t$, then

$$\alpha y = \frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \alpha\frac{dy}{du}.$$

So $y = \dfrac{dy}{du}$ and

$$y = ce^u = ce^{\alpha t},$$

for some constant c.

Assume that a quantity y varies with time t. Then y is said to *grow or decay exponentially* if $y' = \alpha y$, for some constant α.

In this case we know that

$$y = ce^u = ce^{\alpha t},$$

for some constant c.

If $\alpha > 0$ (the growth constant), then y increases exponentially with time.

If $\alpha < 0$ (the decay constant), then y decreases exponentially with time.

If y_0 is the value of y at $t = 0,$ then

$$y_0 = ce^0 = c,$$

$$y = y_0 e^{\alpha t}$$

4.12 Examples

(i) If the decay constant of a radioactive substance is $K < 0,$ then compute the time T (known as the *half-life*) after which only half of the any original quantity remains. At time $t = T:$

$$\frac{1}{2} y_0 = y_0 e^{KT}$$

$$\frac{1}{2} = e^{KT}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{KT}\right) = KT$$

$$-\ln(2) = KT$$

$$-\frac{\ln(2)}{K} = T.$$

(ii) The half-life of radium is 1690 years. If 10% of an original quantity of radium remains, how long ago was the radium created.

From the previous example we know that $1690 = T = -\frac{\ln(2)}{K}$ where K is the decay constant and so

$$K = -\frac{\ln(2)}{1690}.$$

We set $t = 0$ when the radium was created. At the present time t

$$y(t) = y_0 e^{Kt} = 0.1 y_0$$

$$\Rightarrow \quad Kt = \ln(0.1) = -\ln(10)$$

$$\Rightarrow \quad t = -\frac{\ln(10)}{K} = 1690 \frac{\ln(10)}{\ln(2)} = 1690(3.32) \approx 5611$$

So the radium was created approx 5611 years ago.

(iii) A quantity y is said to *grow exponentially* at the rate of r percent per year if

$$y = y_0 \left(1 + \frac{r}{100}\right)^t$$

with t in years. Find the exponential growth constant K for such a quantity.

$$y = y_0 \left(1 + \frac{r}{100}\right)^t = y_0 e^{t \ln\left(1+\frac{r}{100}\right)}.$$

Therefore

$$K = \ln\left(1 + \frac{r}{100}\right).$$

(iv) Under continuous compounding of interest, an invested amount will grow exponentially. What is the initial sum that will be multiplied by five in 8 years and will amount to $E10,000$ after 24 years.

Let P_0 be the initial sum invested, $P(t)$ the sum after t years and r the interest rate. Then

$$P(t) = P_0 e^{\frac{rt}{100}}.$$

We know that $P(8) = 5P_0$ so

$$\begin{aligned} & 5P_0 = P(8) = P_0 e^{\frac{8r}{100}} \\ \Rightarrow \quad & 5 = e^{\frac{8r}{100}} \\ \Rightarrow \quad & \ln(5) = \frac{8r}{100} \\ \Rightarrow \quad & \frac{25}{2}\ln(5) = r \end{aligned}$$

We also know that $P(24) = 10,000$ so that

$$\begin{aligned} & 10,000 = P_0 e^{\frac{24r}{100}} = P_0 e^{\frac{6r}{25}} \\ & = P_0 e^{3\ln(5)} = 125 P_0 \\ \Rightarrow \quad & P_0 = \frac{10,000}{125} = 80. \end{aligned}$$

4.13 Some facts about the exponential function

(i) $e^x > 0,$ for all $x \in \mathbb{R}$ (by definition).

(ii) e^x is increasing as $\frac{d}{dx}(e^x) = e^x > 0.$

(iii) $\frac{d}{dx}(y - \ln(y)) = 1 - \frac{1}{y} > 0$ so that $y - \ln(y)$ is increasing and positive, for $y > 1.$ If $0 < y \leq 1,$ then $y - \ln(y) > y > 0$.

Therefore $x = \ln(e^x) < e^x$ and $e^x - x > 0.$

(iv)

$$\lim_{x\to\infty} e^x > \lim_{x\to\infty} x = \infty.$$

(v) Set $u = -x,$ then

$$\lim_{x\to-\infty} e^x = \lim_{u\to\infty} e^{-u} = \lim_{u\to\infty} \frac{1}{e^u} = 0.$$

4.14 Graphs of exp and ln

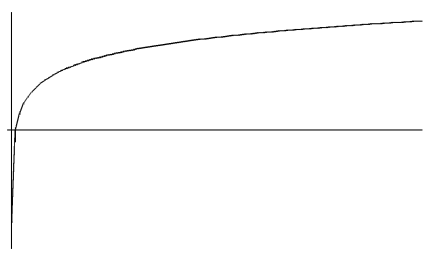

Figure 2: Graph of $f(x) = \ln(x)$, for $x > 0.$

$\ln(x) > 0$ if $x > 1$, $\ln(x) = 0$ if $x = 1$, $\ln(x) < 0$ if $x < 1$.

$\ln$ is an increasing function.

$\lim_{x\to\infty} \ln(x) = \infty.$

$\lim_{x\to0^+} \ln(x) = -\infty.$

$e^x > 0,$ for all $x \in \mathbb{R}.$

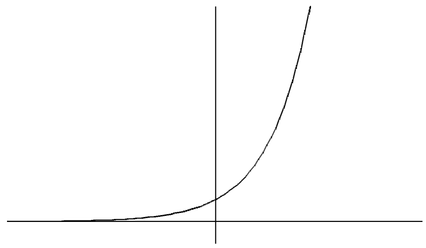

Figure 3: Graph of $f(x) = e^x$.

$\exp$ is an increasing function.

$\lim_{x\to\infty} e^x = \infty$.

$\lim_{x\to-\infty} e^x = 0$.

4.15 L'Hopital's Rule

If $f(x)$ and $g(x)$ both approach 0 or both approach $\pm\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

Here "$\lim$" stands for any of

$$\lim_{x\to\infty}, \lim_{x\to-\infty}, \lim_{x\to a}, \lim_{x\uparrow a}, \lim_{x\downarrow a}.$$

4.16 Example

(i) Find $\lim_{t\to 1} \frac{t^2+t-2}{t^2-1}$.

Here $f(t) = t^2 + t - 2$, $g(t) = t^2 - 1$ and $f(1) = 0 = g(1)$.

Using L'Hopital's Rule

$$\lim_{t\to 1} \frac{t^2+t-2}{t^2-1} = \lim_{t\to 1} \frac{2t+1}{2t} = \frac{3}{2}.$$

(ii) Find $\lim_{t\to -1} \frac{t^9+1}{t^7+1}$. Here $f(t) = t^9 + 1$, $g(t) = t^7 + 1$ and $f(-1) = 0 = g(-1)$.
Using L'Hopital's Rule

$$\lim_{t\to -1} \frac{t^9+1}{t^7+1} = \lim_{t\to -1} \frac{9t^8}{7t^6} = \frac{9}{7}.$$

(iii) Find $\lim_{x\to\infty} \frac{\ln(x)}{x}$.

Here $f(x) = \ln(x)$, $g(x) = x$ and $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$.

Using L'Hopital's Rule

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

(iv) Find $\lim_{x\to\infty} \frac{x^n}{e^x}$, for any positive integer n. Here $f(x) = x^n$, $g(x) = e^x$ and $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$.

Using L'Hopital's Rule

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x}.$$

We can show, by induction that

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = n! \lim_{x\to\infty} \frac{1}{e^x} = 0.$$