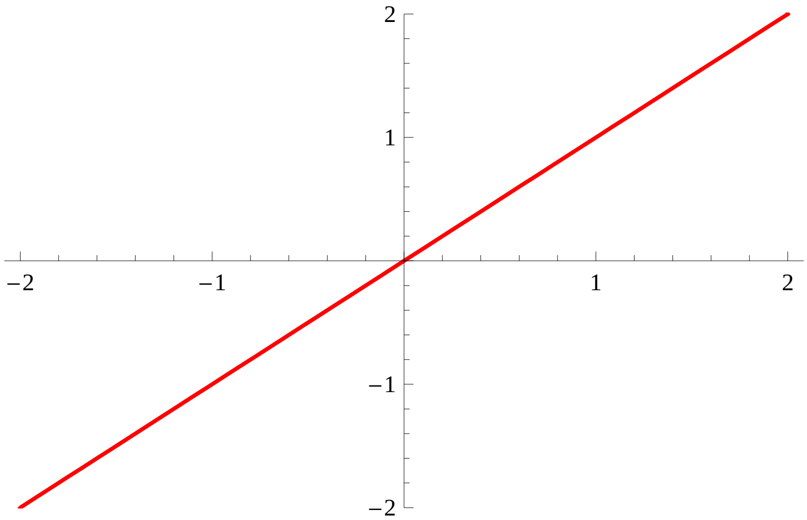
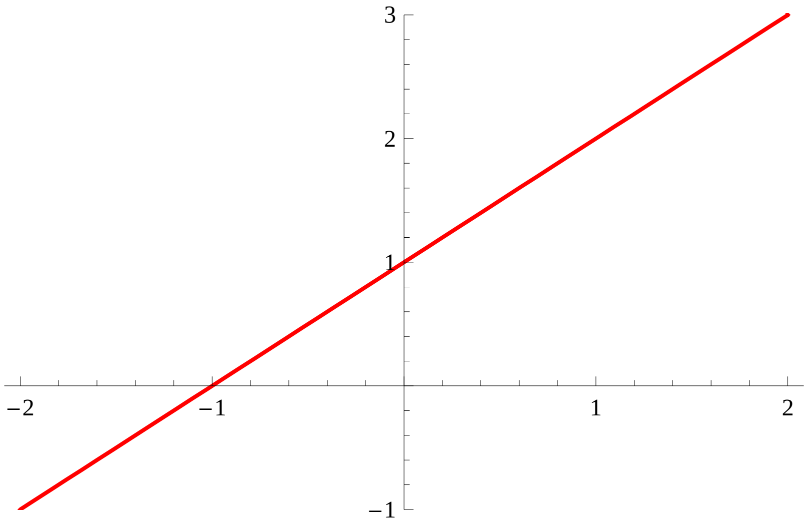
How to graph polynomials

As seen in class, the curve $y = x$ is just the straight line through the origin:



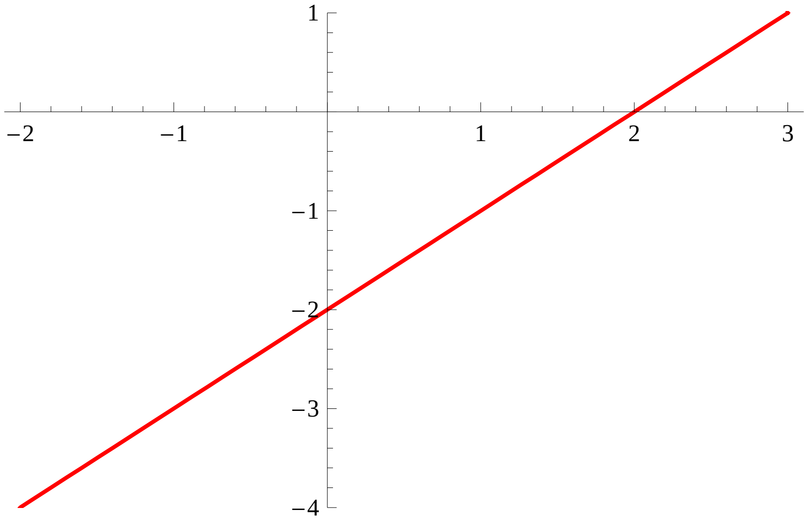
The curve $y = x$

and if we add 1 to the right hand side (to get $y = x + 1$) we just raise the curve $y = x$ by one unit to get the curve:



The curve $y = x + 1$

Similarly, the the curve $y = x - 2$ is just the the curve $y = x$ lowered by two units:



The curve $y = x - 2$

In general, (as was shown in class) the line in the xy-plane which passes through the points

$$P = (x_1, y_1) \quad \text{and} \quad Q = (x_2, y_2), \quad \text{where} \quad x_2 \neq x_1,$$

has equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,.$$

The quantity $(y_2 - y_1)/(x_2 - x_1)$ is called the slope of the line and is denoted by m. Thus we can write

$$y = m\,x + c \quad \text{where} \quad c = y_1 - mx_1\,.$$

In the special case, where $x_2 = x_1$ the line is vertical and has equation $x = x_1$

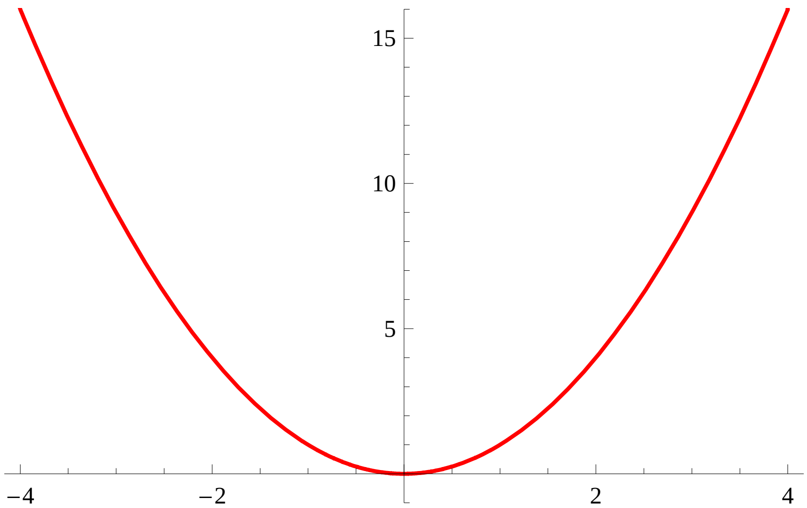
Example [1]: Determine the equation of the line which passes through the points

$$P = (-1,\, 3) \quad \text{and} \quad Q = (2,\, -4).$$

Solution :

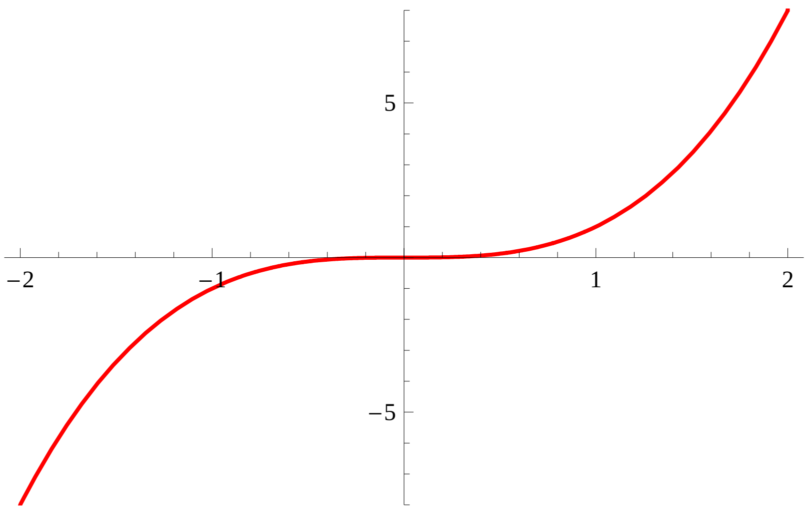
$$\begin{aligned} y - 3 &= \frac{-4-3}{2+1}(x+1) \\ \Longrightarrow \quad y - 3 &= \frac{-7}{3}(x+1) \\ \Longrightarrow \quad 3\,y - 9 &= -7\,x - 7 \\ \Longrightarrow \quad 3\,y &= -7\,x + 2\,. \end{aligned}$$

To plot the curve $y = x^2$, we note that for all $x \in \mathbb{R}$ the quantity $f(x) = x^2$ satisfies (i) $f(x) \geq 0$ and (ii) $f(-x) = f(x)$. Furthermore, when x is positive, $f(x) = x^2$ increases with x. Thus the curve $y = x^2$ is of the form:



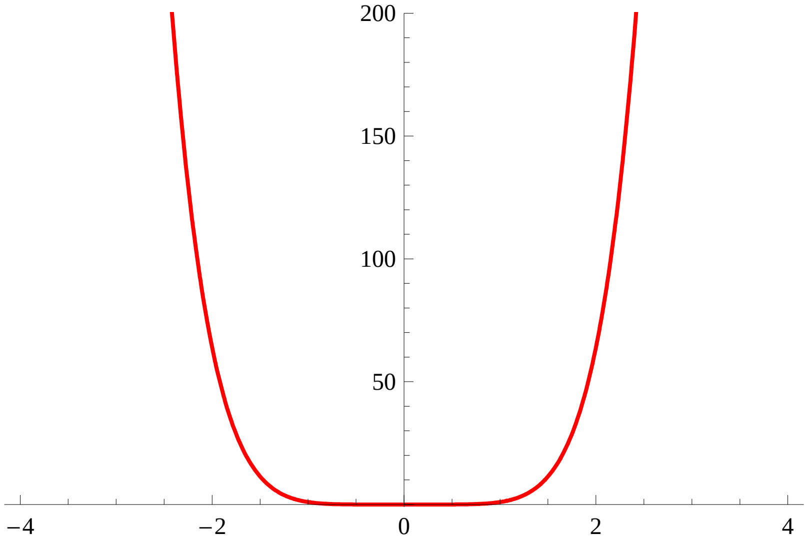
The curve $y = x^2$

Similarly, to plot the curve $y = x^3$, we note that for all $x \in [0, \infty)$ the quantity $f(x) = x^3$ satisfies (i) $f(x) \geq 0$ and (ii) $f(-x) = -f(x)$. Furthermore, when x is positive, $f(x) = x^3$ increases with x. Thus the curve $y = x^3$ is of the form:



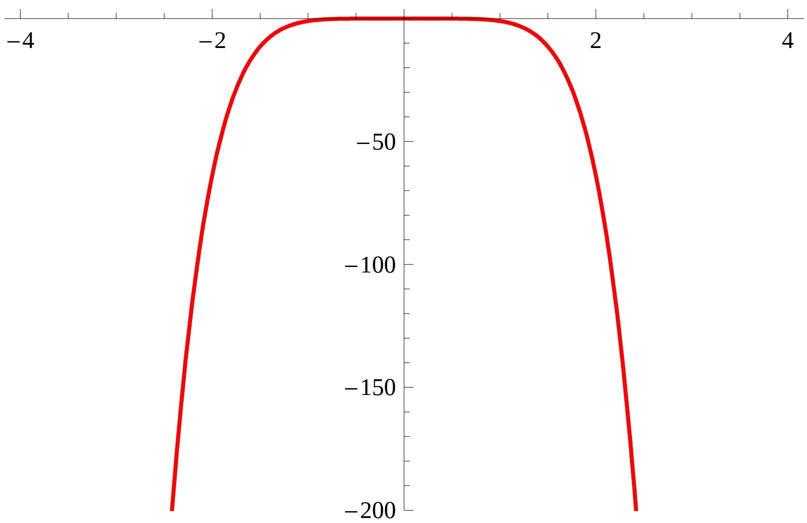
The curve $y = x^3$

In general, to plot the curve $y = x^{\text{even}}$, that is x raised to an even power, we argue as we did in the case of $y = x^2$. Again, for all $x \in \mathbb{R}$ the quantity $f(x) = x^{\text{even}}$ satisfies (i) $f(x) \geq 0$ and (ii) $f(-x) = f(x)$. Furthermore, when x is positive, $f(x) = x^{\text{even}}$ increases with x. Thus the curve $y = x^{\text{even}}$ is of the form:



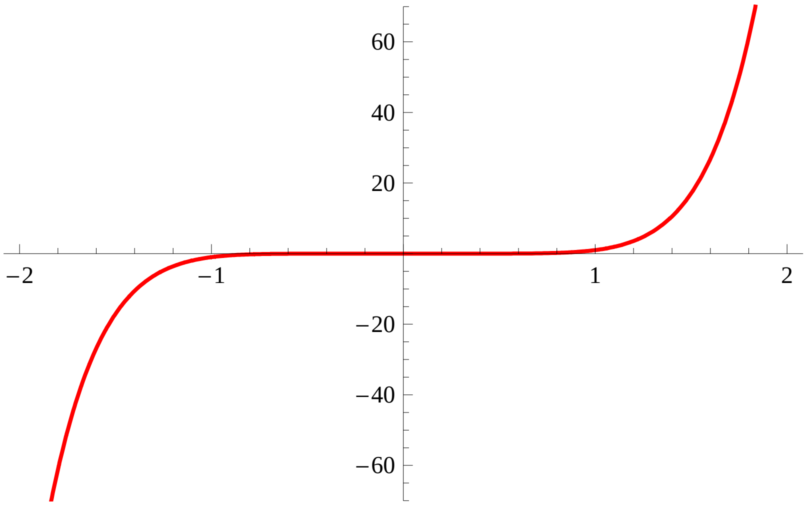
The curve $y = x^{\text{even}}$

Important Remark: The graph of $y = -\,p(x)$ is obtained by reflecting the graph of $y = p(x)$ in the x-axis. Thus, for example, when we reflect the curve $y = x^{\text{even}}$ (given above) in the x-axis we get the curve $y = -\,x^{\text{even}}$ as follows:



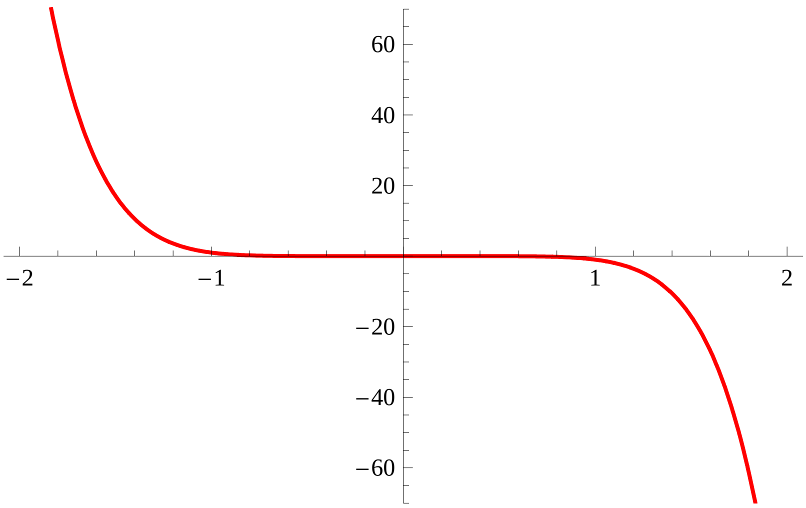
The curve $y = -\,x^{\text{even}}$

The case of an odd power is similar to that of $y = x^3$. Thus to plot the curve $y = x^{\text{odd}}$, we note that for all $x \in [0, \infty)$ the quantity $f(x) = x^{\text{odd}}$ satisfies (i) $f(x) \geq 0$ and (ii) $f(-x) = -f(x)$. Furthermore, when x is positive, $f(x) = x^{\text{odd}}$ increases with x. Thus the curve $y = x^{\text{odd}}$ is of the form:



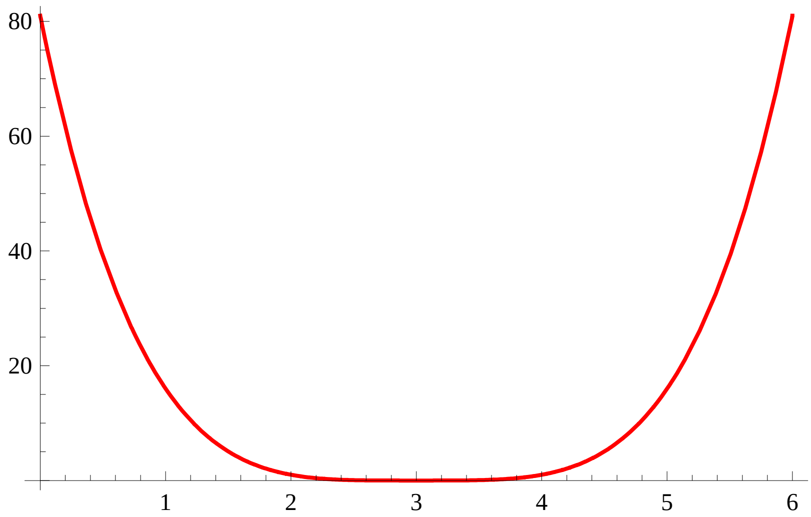
The curve $y = x^{\text{odd}}$

As explained above, by reflecting the curve $y = x^{\text{odd}}$ (given above) in the x-axis we get the curve $y = -\,x^{\text{odd}}$ as follows:



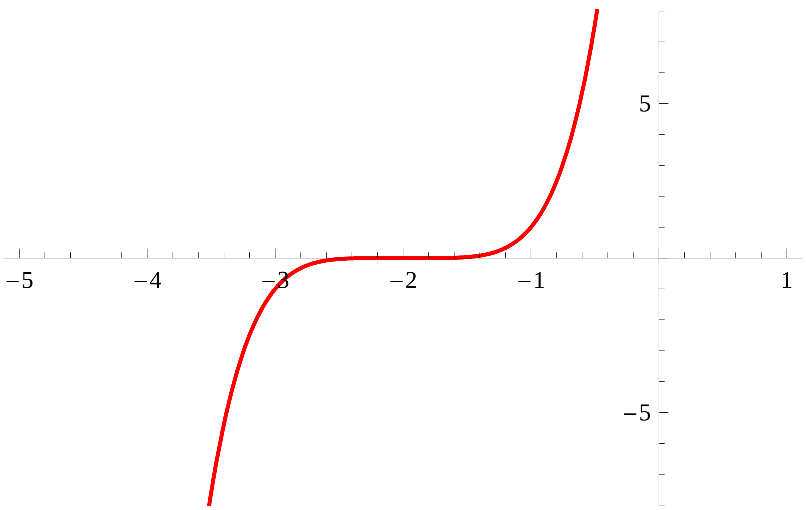
The curve $y = -\,x^{\text{odd}}$

The curve $y = (x - a)^{\text{even}}$, where a is a constant, is similar to the curve $y = x^{\text{even}}$ except that this (latter) curve is translated horizontally a distance a from the origin. For example, below is the curve $y = (x - 3)^{\text{even}}$



The curve $y = (x - 3)^{even}$

Similarly, the curve $y = (x - a)^{\text{odd}}$, where a is a constant, is obtained by translating horizontally the curve $y = x^{\text{odd}}$ a distance a from the origin. For example, below is the curve $y = (x + 2)^{\text{odd}}$:



The curve $y = (x + 2)^{\text{odd}}$

How to sketch the graph of a factorized polynomial: In order to know where to start with the sketch the following is important:

FACT: The behaviour of the polynomial $p(x) = a_n\, x^n + a_{n-1}\, x^{n-1} + \ldots + a_1\, x + a_0$ as $x \longrightarrow -\infty$ or as $x \longrightarrow \infty$ is the same as that of the leading term $a_n\, x^n$.

Note 1: In the case where $a_n > 0$ we have that,

(i) $\quad$ (as $x \longrightarrow -\infty$) $\quad a_n\, x^n \longrightarrow \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$

(ii) $\quad$ (as $x \longrightarrow \infty$) $\quad a_n\, x^n \longrightarrow \infty$ $\quad$ for all n.

Note 2: In particular, if $a_n > 0$ and $n =$ degree of $p(x)$ is odd, then (with x a large negative number) we start drawing the curve $y = p(x)$ with y below the x-axis.

Note 3: If $a_n > 0$ and $n =$ degree of $p(x)$ is even, then (with x a large negative number) we start drawing the curve $y = p(x)$ with y above the x-axis.

Note 4: If $a_n > 0$ and irrespective of whether or not $n =$ degree of $p(x)$ is even or odd, then (with x a large positive number) we finish drawing the curve $y = p(x)$ with y above the x-axis.

We will now illustrate the ideas explained with a series of examples

Example [2]: Sketch the curve

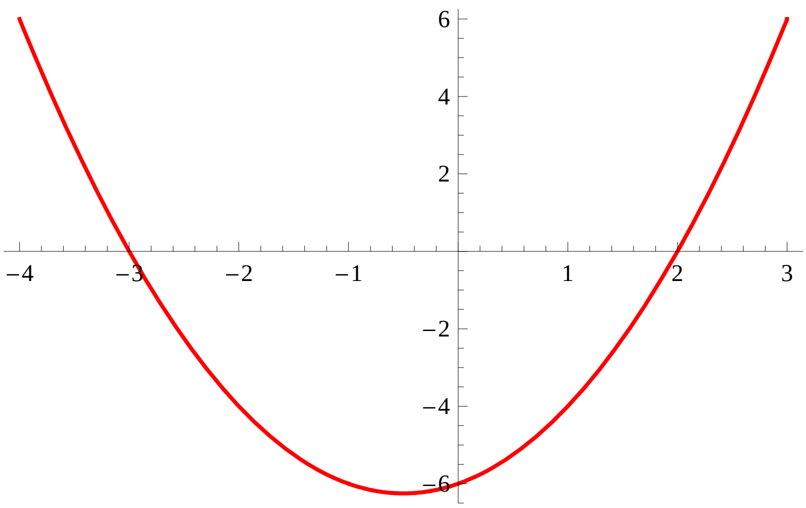
$$y = p(x) = (x+3)(x-2)\,.$$

Here the $\deg[p(x)] = 2 =$ even so (with x a large negative number) we start drawing the curve $y = p(x)$ with y above the x-axis. Observe that $y = 0$ when $x = -3$ or when $x = 2$.

Near $x = -3$ **:** Here $(x-2)$ is near (-5) so that $y = (x+3)(x-2)$ is near $y = -5\,(x+3)$. Therefore, the curve $y = p(x)$ crosses the x-axis, at $x = -3$ in a fashion similar to that of the line $y = -5\,(x+3)$.

Near $x = 2$ **:** Here $(x+3)$ is near (5) so that $y = (x+3)(x-2)$ is near $y = 5\,(x-2)$. Therefore, the curve $y = p(x)$ crosses the x-axis, at $x = 2$ in a fashion similar to that of the line $y = 5\,(x+2)$.

Putting all of this together we see that the curve $y = (x+3)(x-2)$ is of the form:



The curve $y = (x+3)(x-2)$

Example [3]: Sketch the curve

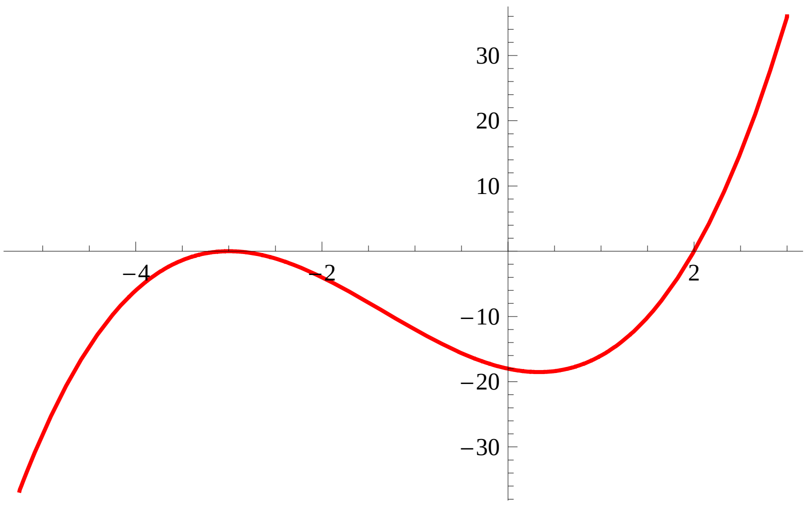
$$y = p(x) = (x+3)^2(x-2)\,.$$

Here the $\deg[p(x)] = 3 =$ odd so (with x a large negative number) we start drawing the curve $y = p(x)$ with y below the x-axis. Observe that again $y = 0$ when $x = -3$ or when $x = 2$.

Near $x = -3$: Here $(x-2)$ is near (-5) so that $y = (x+3)^2(x-2)$ is near $y = -5\,(x+3)^2$. Therefore, the curve $y = p(x)$ just touches (" or bounces off ") the x-axis, at $x = -3$ in a fashion similar to that of the line $y = -5\,(x+3)^2$.

Near $x = 2$: Here $(x+3)^2$ is near (25) so that $y = (x+3)^2(x-2)$ is near $y = 25\,(x-2)$. Therefore, the curve $y = p(x)$ crosses the x-axis, at $x = 2$ in a fashion similar to that of the line $y = 25\,(x+2)$.

Putting all of this together we see that the curve $y = (x+3)^2(x-2)$ is of the form:



The curve $y = (x+3)^2(x-2)$

Example [4]: Sketch the curve

$$y = p(x) = (x+3)^2(x+1)(x-2)^2 \,.$$

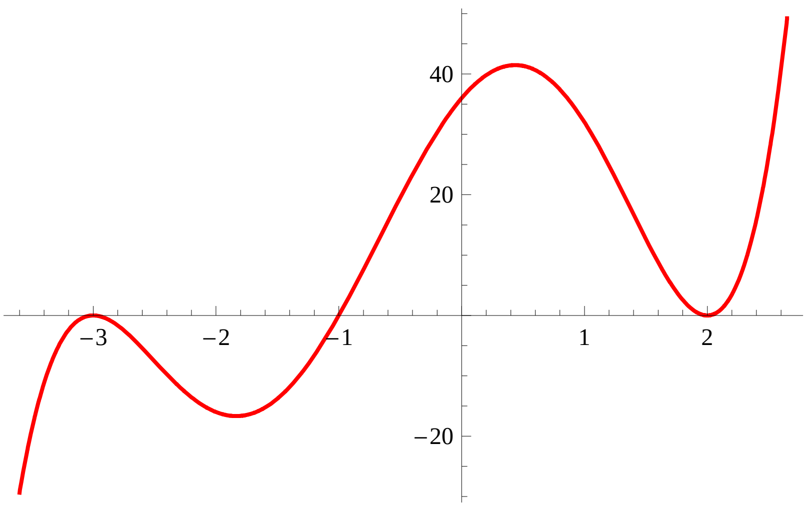
Here the $\deg[p(x)] = 5 =$ odd so (with x a large negative number) we start drawing the curve $y = p(x)$ with y below the x-axis. Observe that now $y = 0$ when $x = -3$, $x = -1$ or when $x = 2$.

Near $x = -3$: Here $(x+1)(x-2)^2$ is near some constant $= K_1$, say, so that $y = (x+3)^2(x+1)(x-2)^2$ is near $y = K_1\,(x+3)^2$. Therefore, the curve $y = p(x)$ just touches (" or bounces off ") the x-axis, at $x = -3$ in a fashion similar to that of the curve $y = K_1\,(x+3)^2$.

Near $x = -1$: Here $(x+3)^2(x-2)^2$ is near some constant $= K_2$, say, so that $y = (x+3)^2(x+1)(x-2)^2$ is near $y = K_2\,(x+1)$. Therefore, the curve $y = p(x)$ crosses the x-axis, at $x = -1$ in a fashion similar to that of the line $y = K_2\,(x+1)$.

Near $x = 2$: Here $(x+3)^2(x+1)$ is near some constant $= K_3$, say, so that $y = (x+3)^2(x+1)(x-2)^2$ is near $y = = K_3\,(x-2)^2$. Therefore, the curve $y = p(x)$ just touches (" or bounces off ") the x-axis, at $x = 2$ in a fashion similar to that of the curve $y = = K_3\,(x-2)^2$.

Putting all of this together we see that the curve $y = (x+3)^2(x+1)(x-2)^2$ is of the form:



The curve $y = (x+3)^2(x+1)(x-2)^2$

Example [5]: Sketch the curve

$$y = p(x) = (x+3)^3(x+1)(x-2)^2 \,.$$

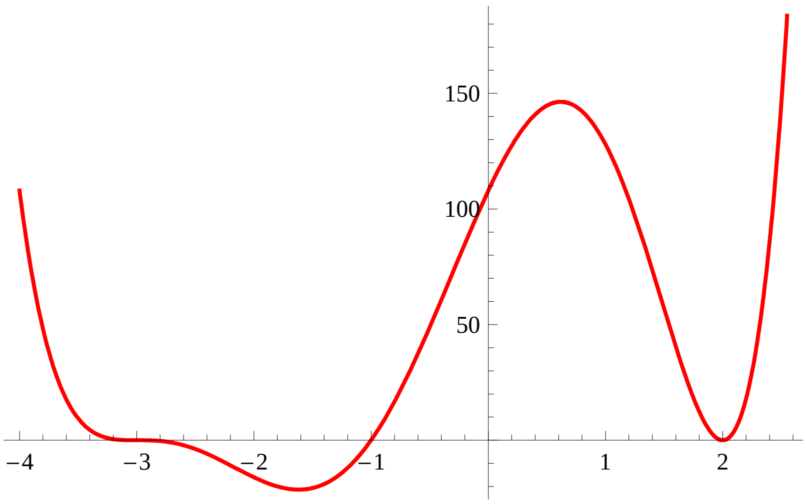
Here the $\deg[p(x)] = 6 =$ even so (with x a large negative number) we start drawing the curve $y = p(x)$ with y above the x-axis. Observe that again $y = 0$ when $x = -3$, $x = -1$ or when $x = 2$. Now, you go through the various cases yourself, that is, the cases:

Near $x = -3$:

Near $x = -1$:

Near $x = 2$:

and convince yourself that the curve $y = (x+3)^3(x+1)(x-2)^2$ is of the form:



The curve $y = (x+3)^3(x+1)(x-2)^2$

More examples will be given in class.