# Rational Roots of $\mathbb{Z}$-Coefficient Polynomials

In this note we will consider only those polynomials

$$p(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

where the coefficients $a_n,\ a_{n-1}\ \ldots\ a_1,\ a_0$ are all integers. For example, the coefficients of the polynomial

$$p(x) = 30\,x^3 - 13\,x^2 - 13\,x + 6$$

are all integers.

We wish to find all the **rational roots** of such a polynomial

$$p(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0.$$

That is we wish to find all

$$\frac{\ell}{m} \in \mathbb{Q} \quad \text{such that} \quad p\left(\frac{\ell}{m}\right) = 0.$$

After cancelation of all common factors (if there are any to begin with) we can assume that the **integers** $\ell$ and $m$ have **no common factors**. Now,

$$0 = p\left(\frac{\ell}{m}\right)$$

$$\Longrightarrow \quad 0 = a_n\left(\frac{\ell}{m}\right)^n + a_{n-1}\left(\frac{\ell}{m}\right)^{n-1} + \cdots + a_1\left(\frac{\ell}{m}\right) + a_0$$

Now, multiply across this latter equation by $m^n$.

$$\Longrightarrow \quad 0 = a_n\,\ell^n + a_{n-1}\,\ell^{n-1}\,m^1 + \cdots + a_1\,\ell\,m^{n-1} + a_0\,m^n$$

$$\Longrightarrow \quad -a_0\,m^n = a_n\,\ell^n + a_{n-1}\,\ell^{n-1}\,m^1 + \cdots + a_1\,\ell\,m^{n-1}$$

$$\Longrightarrow \quad -a_0\,m^n = \ell\left[a_n\,\ell^{n-1} + a_{n-1}\,\ell^{n-2}\,m^1 + \cdots + a_1\,m^{n-1}\right]$$

$$\Longrightarrow \quad a_0\,m^n = \ell\,[\,\text{some integer}\,]$$

Thus, we have shown that:

$$\boxed{p\left(\frac{\ell}{m}\right) = 0} \quad \Longrightarrow \quad \boxed{a_0\, m^n = \ell\,[\text{some integer}]}$$

$$\Longrightarrow \quad \boxed{\ell \text{ is a factor of } (a_0\, m^n)}$$

$$\Longrightarrow \quad \boxed{\begin{array}{c} \ell \text{ is a factor of } a_0\,, \\ \text{because the integers } \ell \text{ and } m \\ \text{have } \textbf{no common factors}. \end{array}}$$

A slight modification of the above argument (you should provide the details) shows, in addition, that:

$$\boxed{p\left(\frac{\ell}{m}\right) = 0} \quad \Longrightarrow \quad \boxed{m \text{ is a factor of } a_n}\ .$$

Finally, putting these two statements together, we have shown that:

$$\boxed{\boxed{p\left(\frac{\ell}{m}\right) = 0} \quad \Longrightarrow \quad \boxed{\begin{array}{c} \ell \text{ is a factor of } a_0 \text{ and} \\ m \text{ is a factor of } a_n \end{array}}}$$

**Example:** Find all the **rational roots** of the polynomial:

$$p(x) = 30\,x^3 - 13\,x^2 - 13\,x + 6$$

**Solution:** In this example $n = 3$, $a_3 = 30$ and $a_0 = 6$. According to the statement above, it follows that

$$\boxed{p\left(\frac{\ell}{m}\right) = 0} \quad \Longrightarrow \quad \boxed{\begin{array}{c} \ell \text{ is a factor of } 6 \text{ and} \\ m \text{ is a factor of } 30 \end{array}}\ .$$

Thus, $\ell \in \{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6\}$ and $m \in \{\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30\}$.

Now, after choosing in turn, $\ell$ from the set $\{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6\}$ and $m$ from the set $\{\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30\}$ and after eliminating all repetitions we find that:

$$\frac{\ell}{m} \in \left\{ \begin{array}{l} \pm 1,\ \pm 2,\ \pm 3,\ \pm 6, \\ \pm\frac{1}{2},\ \pm\frac{3}{2}, \\ \pm\frac{1}{3},\ \pm\frac{2}{3}, \\ \pm\frac{1}{5},\ \pm\frac{2}{5},\ \pm\frac{3}{5},\ \pm\frac{6}{5}, \\ \pm\frac{1}{6},\ \pm\frac{5}{6}, \\ \pm\frac{1}{10},\ \pm\frac{3}{10}, \\ \pm\frac{1}{15},\ \pm\frac{2}{15}, \\ \pm\frac{1}{30} \end{array} \right\}$$

We are now in the happy situation of knowing that if the polynomial

$$p(x) = 30\,x^3 - 13\,x^2 - 13\,x + 6$$

has any **rational roots**, then they must lie in the above set. We begin to evaluate $p(x)$ at each $x$, in turn, in the above set to find that:

$$p(\pm 1) \neq 0, \qquad p(\pm 2) \neq 0, \qquad p(\pm 3) \neq 0, \qquad p(\pm 6) \neq 0, \qquad \text{but} \qquad p\left(\frac{1}{2}\right) = 0.$$

That is $1/2$ is a **root** of $p(x)$ and, accordingly:

$$\boxed{\frac{1}{2} \text{ being a root of } p(x)} \implies \boxed{\left(x - \frac{1}{2}\right) \text{ is a factor of } p(x)}$$

$$\implies \boxed{(\,2\,x - 1\,) \text{ is a factor of } p(x)}$$

After long division of $p(x)$ by $(2x - 1)$ we find that

$$p(x) = (2x - 1)(15x^2 + x - 6).$$

We could repeat the above process on the factor $15x^2 + x - 6$, and indeed, this is exactly what we would in the case where the original polynomial $p(x)$ has degree greater than 3. However, in this case the factor $15x^2 + x - 6$ is just a quadratic which you know how to factorize from school. In fact, it is easy to see that:

$$15x^2 + x - 6 = (3x + 2)(5x - 3)$$

and, therefore,

$$p(x) = (2x - 1)(3x + 2)(5x - 3).$$

Finally, the roots of $p(x) = 30x^3 - 13x^2 - 13x + 6$ are:

$$x = \frac{1}{2}, \qquad x = -\frac{2}{3} \quad \text{and} \quad x = \frac{3}{5}.$$