

# RELATIONS 1

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What we wish to discuss (and make precise) here is that vague notion we have of two (or more) things being related to one another.

**Example [1] (“Being the owner of”)** : Here we will say that

$$\boxed{\text{object } a \text{ is related to object } b} \iff \boxed{a \text{ owns } b}.$$

We might wish to be a bit more precise and specify what kind of object is  $a$ . For example, is it a person? Is it a corporation? Or is it perhaps something else? Likewise, we might want to be more specific about what type of object is  $b$ .

To make all of this precise we begin by specifying two sets  $A$  and  $B$  so that

**the owners are elements of  $A$**

and

**the objects which are owned are elements of  $B$ .**

Next, we form what’s called the Cartesian product of  $A$  and  $B$ , this is denoted by  $A \times B$ , and, finally, (by definition) a relation from set  $A$  to set  $B$  is just a subset of  $A \times B$ . We now formalise these in the following:

**Definition (The Cartesian Product)** : Given any two sets  $A$  and  $B$ , we define

$$A \times B = \{ \text{ordered pairs } (a, b) \mid a \in A \text{ and } b \in B \},$$

called the the Cartesian product of  $A$  and  $B$ .

Diagram 1.

**Definition (A Relation)** : A relation,  $\mathfrak{R}$ , from a set  $A$  to a set  $B$  is just a subset of  $A \times B$  and, for any  $(a, b) \in A \times B$ , we say that

$$a \text{ is related to } b \iff (a, b) \in \mathfrak{R}.$$

In this context, we often write  $a \mathfrak{R} b$  to mean that  $(a, b) \in \mathfrak{R}$  simply because  $a \mathfrak{R} b$  is a convenient abbreviation of the phrase  **$a$  is related to  $b$** .

Diagram 2.

**Example [2] (“Is the father of”)** : Here we could take

$$A = \{ \text{all living human males} \} \quad \text{and} \quad B = \{ \text{all living humans} \}$$

and we will say that

$$\boxed{a \mathfrak{R} b} \iff \boxed{\text{a is the father of } b}.$$

Diagram 3.

**Example [3]** (“ Is less than or equal to ”) : Here we could take

$$A = \mathbb{R} \quad \text{and} \quad B = \mathbb{R}$$

and we will say that

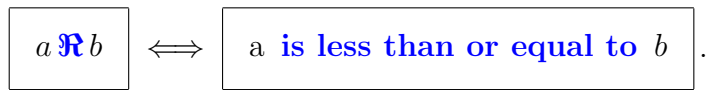


Diagram 4.

**Example [4]** (“ Was born in the same year as ”) : Here we could take

$$A = \{ \text{all living humans} \} \quad \text{and} \quad B = \{ \text{all living humans} \}$$

and we will say that

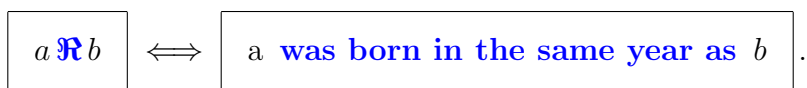


Diagram 5.

**Example [5]** (“**Has a factor in common with**”) : Here we could take

$$A = B = \mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \dots\}$$

and we will say that

$$\boxed{a \mathfrak{R} b} \iff \boxed{\text{a has a factor in common with } b}.$$

**Note :** By a common factor here we mean a common factor which is bigger than 1.

Diagram 6.

**Relations on a set and their properties :** By definition,

a **binary relation on a set**  $A$  is just a subset of  $A \times A$ .

Relations of this type are extremely important especially when they satisfy additional properties which we now describe:

**Definition (Properties of a Relation) :** We say that a relation,  $\mathfrak{R}$ , on a set  $A$  is:

- (i) **Reflexive** if and only if  $a \mathfrak{R} a$  for all  $a \in A$ .
- (ii) **Symmetric** if and only if

$$\boxed{a \mathfrak{R} b} \implies \boxed{b \mathfrak{R} a} \text{ for all } a, b \in A.$$

- (iii) **Anti-Symmetric** if and only if

$$\boxed{a \mathfrak{R} b \text{ and } b \mathfrak{R} a} \implies \boxed{a = b} \text{ for all } a, b \in A.$$

- (iv) **Transitive** if and only if

$$\boxed{a \mathfrak{R} b \text{ and } b \mathfrak{R} c} \implies \boxed{a \mathfrak{R} c} \text{ for all } a, b \text{ and } c \in A.$$

Let's check the examples given above to see which properties, if any, they satisfy. To begin with Examples [1] and [2] satisfy none of these properties since the relations here are not on the same set, that is  $B \neq A$  in these two examples.

**Example [3]** :  $A = B = \mathbb{R}$  and by definition

$$\boxed{a \mathfrak{R} b} \iff \boxed{\text{a is less than or equal to } b}.$$

We now check, in turn, each of the properties:

(i)  $\mathfrak{R}$  is **Reflexive** because  $a \mathfrak{R} a$  for all  $a \in A$ .

(ii)  $\mathfrak{R}$  is **not Symmetric** because, for example

$$\boxed{2 \leq 3, \text{ that is } 2 \mathfrak{R} 3} \quad \text{but} \quad \boxed{3 \not\leq 2, \text{ that is } 3 \not\mathfrak{R} 2}.$$

(iii)  $\mathfrak{R}$  is **Anti-Symmetric** because

$$\boxed{(a \leq b) \text{ and } (b \leq a)} \implies \boxed{a = b} \quad \text{for all } a, b \in A.$$

(iv)  $\mathfrak{R}$  is **Transitive** because

$$\boxed{(a \leq b) \text{ and } (b \leq c)} \implies \boxed{a \leq c} \quad \text{for all } a, b \text{ and } c \in A.$$

**Example [4]** :  $A = B = \{ \text{all living humans} \}$  and by definition

$$\boxed{a \mathfrak{R} b} \iff \boxed{\text{a was born in the same year as } b} .$$

Again we check, in turn, each of the properties:

(i)  $\mathfrak{R}$  is **Reflexive** because, for all  $a \in A$ ,

$$\boxed{a \text{ was born in the same year as } a} .$$

(ii)  $\mathfrak{R}$  is **Symmetric** because, for all  $a, b \in A$ ,

$$\boxed{a \text{ born in the same year as } b} \implies \boxed{b \text{ was born in the same year as } a} .$$

(iii)  $\mathfrak{R}$  is **not Anti-Symmetric** because, for all  $a, b \in A$ ,

$$\boxed{\begin{array}{c} a \text{ born in the same year as } b \\ \text{and} \\ b \text{ born in the same year as } a \end{array}} \not\Rightarrow \boxed{a = b} .$$

(iv)  $\mathfrak{R}$  is **Transitive** because, for all  $a, b$  and  $c \in A$ ,

$$\boxed{\begin{array}{c} a \text{ born in the same year as } b \\ \text{and} \\ b \text{ born in the same year as } c \end{array}} \implies \boxed{a \text{ born in the same year as } c} .$$

**Example [5]** :  $A = B = \mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \dots\}$  and by definition

$$\boxed{a \mathfrak{R} b} \iff \boxed{a \text{ has a factor in common with } b} .$$

Again we check, in turn, each of the properties:

(i)  $\mathfrak{R}$  is **Reflexive** because, for all  $a \in A$ ,

$$\boxed{a \text{ certainly has a factor in common with } a} .$$

(ii)  $\mathfrak{R}$  is **Symmetric** because, for all  $a, b \in A$ ,

$$\boxed{a \text{ has a factor in common with } b} \implies \boxed{b \text{ has a factor in common with } a} .$$

(iii)  $\mathfrak{R}$  is **not Anti-Symmetric** because, for all  $a, b \in A$ ,

$$\boxed{\begin{array}{l} a \text{ having a factor in common with } b \\ \text{and} \\ b \text{ having a factor in common with } a \end{array}} \not\Rightarrow \boxed{a = b} .$$

(iv)  $\mathfrak{R}$  is **not Transitive** because, for all  $a, b$  and  $c \in A$ ,

$$\boxed{\begin{array}{l} a \text{ having a factor in common with } b \\ \text{and} \\ b \text{ having a factor in common with } c \end{array}} \not\Rightarrow \boxed{a \text{ has a factor in common with } c} .$$

For example,

$$\boxed{\begin{array}{l} 4 \text{ has a factor in common with } 6, \text{ namely } 2 \\ \text{and} \\ 6 \text{ has a factor in common with } 9, \text{ namely } 3, \end{array}}$$

but

$$\boxed{4 \text{ does not have a factor in common with } 9} .$$