RELATIONS 1

What we wish to discuss (and make precise) here is that vague notion we have of two (or more) things being related to one another.

Example [1] ("Being the owner of") : Here we will say that

object ***a*** is **related to** object ***b*** $\iff$ ***a*** **owns** ***b*** .

We might wish to be a bit more precise and specify what kind of object is ***a***. For example, is it a person? Is it a corporation? Or is it perhaps something else? Likewise, we might want to be more specific about what type of object is ***b*** .

To make all of this precise we begin by specifying two sets A and B so that

the owners are elements of *A*

and

the objects which are owned are elements of *B*.

Next, we form what's called the Cartesian product of A and B, this is denoted by $A \times B$, and, finally, (by definition) a relation from set A to set B is just a subset of $A \times B$. We now formalise these in the following:

Definition (The Cartesian Product) : Given any two sets A and B, we define

$$A \times B \;=\; \{\, \textbf{ordered pairs } (a,\, b) \;\mid\; a \in A \;\text{ and }\; b \in B \,\},$$

called the the Cartesian product of A and B.

Diagram 1.

Definition (A Relation) : A relation, $\Re$, from a set A to a set B is just a subset of $A \times B$ and, for any $(a,\, b) \in A \times B$, we say that

$$a \text{ is related to } b \iff (a,\, b) \in \Re.$$

In this context, we often write $a \,\Re\, b$ to mean that $(a,\, b) \in \Re$ simply because $\boldsymbol{a \,\Re\, b}$ is a convenient abbreviation of the phrase ***a* is related to *b***.

Diagram 2.

Example [2] (“Is the father of”) : Here we could take

$$A = \{\,\text{all living human males}\,\} \quad \text{and} \quad B = \{\,\text{all living humans}\,\}$$

and we will say that

$$\boxed{a \,\boldsymbol{\Re}\, b} \iff \boxed{\text{a } \textbf{is the father of } b}.$$

Diagram 3.

Example [3] (" Is less than or equal to") : Here we could take

$$A = \mathbb{R} \quad \text{and} \quad B = \mathbb{R}$$

and we will say that

$$\boxed{a\, \mathfrak{R}\, b} \iff \boxed{\text{a } \textbf{is less than or equal to } b}.$$

Diagram 4.

Example [4] (" Was born in the same year as") : Here we could take

$$A = \{\,\text{all living humans}\,\} \quad \text{and} \quad B = \{\,\text{all living humans}\,\}$$

and we will say that

$$\boxed{a\, \mathfrak{R}\, b} \iff \boxed{\text{a } \textbf{was born in the same year as } b}.$$

Diagram 5.

Example [5] (" Has a factor in common with ") : Here we could take

$$A = B = \mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \dots\}$$

and we will say that

$$\boxed{a \,\Re\, b} \iff \boxed{\text{a } \textbf{has a factor in common with } b}.$$

Note : By a common factor here we mean a common factor which is bigger that 1.

Diagram 6.

Relations on a set and their properties : By definition,

a binary relation on a set A is just a subset of $A \times A$.

Relations of this type are extremely important especially when they satisfy additional properties which we now describe:

Definition (Properties of a Relation) : We say that a relation, $\Re$, on a set A is:

(i) **Reflexive** if and only if $a \,\Re\, a$ for all $a \in A$.

(ii) **Symmetric** if and only if

$$\boxed{a \,\Re\, b} \implies \boxed{b \,\Re\, a} \quad \text{for all } a, b \in A.$$

(iii) **Anti-Symmetric** if and only if

$$\boxed{a \,\Re\, b \textbf{ and } b \,\Re\, a} \implies \boxed{a = b} \quad \text{for all } a, b \in A.$$

(iv) **Transitive** if and only if

$$\boxed{a \,\Re\, b \textbf{ and } b \,\Re\, c} \implies \boxed{a \,\Re\, c} \quad \text{for all } a, b \text{ and } c \in A.$$

Let's check the examples given above to see which properties, if any, they satisfy. To begin with Examples [1] and [2] satisfy none of these properties since the relations here are not on the same set, that is $B \neq A$ in these two examples.

Example [3] : $A = B = \mathbb{R}$ and by definition

$$\boxed{a \, \boldsymbol{\Re} \, b} \iff \boxed{\text{a } \textbf{is less than or equal to } b}.$$

We now check, in turn, each of the properties:

(i) $\Re$ is **Reflexive** because $a \Re a$ for all $a \in A$.

(ii) $\Re$ is **not Symmetric** because, for example

$$\boxed{2 \leq 3, \text{ that is } 2 \Re 3} \text{ but } \boxed{3 \not\leq 2, \text{ that is } 3 \not\Re 2}.$$

(iii) $\Re$ is **Anti-Symmetric** because

$$\boxed{(a \leq b) \textbf{ and } (b \leq a)} \implies \boxed{a = b} \text{ for all } a, b \in A.$$

(iv) $\Re$ is **Transitive** because

$$\boxed{(a \leq b) \textbf{ and } (b \leq c)} \implies \boxed{a \leq c} \text{ for all } a, b \text{ and } c \in A.$$

Example [4] : $A = B = \{\text{all living humans}\}$ and by definition

$$\boxed{a \,\Re\, b} \iff \boxed{\text{a } \textbf{was born in the same year as } b}.$$

Again we check, in turn, each of the properties:

(i) $\Re$ is **Reflexive** because, for all $a \in A$,

$$\boxed{a \text{ was born in the same year as } a}.$$

(ii) $\Re$ is **Symmetric** because, for all $a, b \in A$,

$$\boxed{a \text{ born in the same year as } b} \Longrightarrow \boxed{b \text{ was born in the same year as } a}.$$

(iii) $\Re$ is **not Anti-Symmetric** because, for all $a, b \in A$,

$$\boxed{\begin{array}{c} a \text{ born in the same year as } b \\ \text{and} \\ b \text{ born in the same year as } a \end{array}} \not\Rightarrow \boxed{a = b}.$$

(iv) $\Re$ is **Transitive** because, for all a, b and $c \in A$,

$$\boxed{\begin{array}{c} a \text{ born in the same year as } b \\ \text{and} \\ b \text{ born in the same year as } c \end{array}} \Longrightarrow \boxed{a \text{ born in the same year as } c}.$$

Example [5] : $A = B = \mathbb{N} \setminus \{1\} = \{2, 3, 4, 5, \ldots\}$ and by definition

$$\boxed{a \,\Re\, b} \iff \boxed{a \text{ has a factor in common with } b}.$$

Again we check, in turn, each of the properties:

(i) $\Re$ is **Reflexive** because, for all $a \in A$,

$$\boxed{a \text{ certainly has a factor in common with } a}.$$

(ii) $\Re$ is **Symmetric** because, for all $a, b \in A$,

$$\boxed{a \text{ has a factor in common with } b} \Longrightarrow \boxed{b \text{ has a factor in common with } a}.$$

(iii) $\Re$ is **not Anti-Symmetric** because, for all $a, b \in A$,

$$\boxed{\begin{array}{c} a \text{ having a factor in common with } b \\ \text{and} \\ b \text{ having a factor in common with } a \end{array}} \not\Rightarrow \boxed{a = b}.$$

(iv) $\Re$ is **not Transitive** because, for all a, b and $c \in A$,

$$\boxed{\begin{array}{c} a \text{ having a factor in common with } b \\ \text{and} \\ b \text{ having a factor in common with } c \end{array}} \not\Rightarrow \boxed{a \text{ has a factor in common with } c}.$$

For example,

$$\boxed{\begin{array}{c} \mathbf{4} \text{ has a factor in common with } 6, \quad \text{namely } 2 \\ \text{and} \\ 6 \text{ has a factor in common with } \mathbf{9}, \quad \text{namely } 3, \end{array}}$$

but

$$\boxed{\mathbf{4} \text{ does not have a factor in common with } \mathbf{9}}.$$