RELATIONS 2

Among all relations, there are two types which are of particular importance, namely; equivalence relations and order relations. In this section we will discuss equivalence relations.

Definition (Equivalence Relation) : We say that a relation, $\Re$, on a set A is an equivalence relation if and only if it is:

(i) **Reflexive.** That is,

$$\boxed{a \,\Re\, a} \quad \text{for all} \quad a \in A\,.$$

(ii) **Symmetric.** That is,

$$\boxed{a \,\Re\, b} \implies \boxed{b \,\Re\, a} \quad \text{for all} \quad a,\, b \in A\,.$$

(iii) **Transitive.** That is,

$$\boxed{a \,\Re\, b \;\textbf{ and }\; b \,\Re\, c} \implies \boxed{a \,\Re\, c} \quad \text{for all} \quad a,\, b \text{ and } c \in A\,.$$

Example [1] (“Has the same mother as”) : Here we take

$$A = \{\text{ all living humans }\}$$

and we will say that

$$\boxed{a\,\Re\,b} \iff \boxed{\text{a \textbf{as the same mother as} } b}.$$

Let's check the properties given above:

(i) **Reflexivity.** For all $a \in A$, it is true that

$$\boxed{a \text{ has the same mother as } a},$$

thus, $\Re$ is reflexive.

(ii) **Symmetry.** For all $a, b \in A$, it is true that

$$\boxed{a \text{ having the same mother as } b} \implies \boxed{b \text{ has the same mother as } a},$$

thus, $\Re$ is symmetric.

(iii) **Transitivity.** For all $a, b, c \in A$, it is true that

$$\boxed{\begin{array}{c} a \text{ having the same mother as } b \\ \text{and} \\ b \text{ having the same mother as } c \end{array}} \implies \boxed{a \text{ has the same mother as } c},$$

thus, $\Re$ is transitive.

Finally, we conclude that this relation $\Re$ is indeed an equivalence relation because it is reflexive, symmetric and transitive.

Example [2] ("After dividing by 7, leaves the same remainder as") : Here we take

$$A = \mathbb{N} \cup \{0\} = \{\,0,\ 1,\ 2,\ 3,\ 4,\ \ldots\,\}$$

and we will say that

$$\boxed{a\,\Re\,b} \iff \boxed{\begin{array}{c} a, \textbf{ after dividing by } 7, \\ \textbf{leaves the same remainder as } b \end{array}}.$$

For example, the numbers 3, 10, 17, 24, ... are all related, because they all leave the remainder 3 when divided by 7. Now, let's check if this is an equivalence relation.

(i) **Reflexivity.** For all $a \in A$, it is true that

$$\boxed{\begin{array}{c} a, \text{ after dividing by } 7, \\ \text{leaves the same remainder as } a \end{array}},$$

thus, $\Re$ is reflexive.

(ii) **Symmetry.** For all $a,\ b \in A$, it is true that

$$\boxed{\begin{array}{c} a, \text{ after dividing by } 7, \\ \text{leaving the} \\ \text{same remainder as } b \end{array}} \Longrightarrow \boxed{\begin{array}{c} b, \text{ after dividing by } 7, \\ \text{leaves the} \\ \text{same remainder as } a \end{array}},$$

thus, $\Re$ is symmetric.

(iii) **Transitivity.** For all $a,\ b,\ c \in A$, it is true that

$$\boxed{\begin{array}{c} \left(\begin{array}{c} a, \text{ after dividing by } 7, \\ \text{leaving the} \\ \text{same remainder as } b \end{array}\right) \\ \\ \text{and} \\ \\ \left(\begin{array}{c} b, \text{ after dividing by } 7, \\ \text{leaving the} \\ \text{same remainder as } c \end{array}\right) \end{array}} \Longrightarrow \boxed{\begin{array}{c} a, \text{ after dividing by } 7, \\ \text{leaving the} \\ \text{same remainder as } c \end{array}}.$$

thus, $\Re$ is transitive.

Again, we conclude that this relation $\Re$ is indeed an equivalence relation because it is reflexive, symmetric and transitive.

Example [3] : Here we take

$$A = \mathbb{R}^2 = \{\, \text{ordered pairs } (x,\, y) \mid x,\, y \in \mathbb{R} \,\}$$

and we will say that

$$\boxed{(x_1,\, y_1)\, \boldsymbol{\Re}\, (x_2,\, y_2)} \iff \boxed{(\, y_1 - y_2\,) \;=\; 3(\, x_1 - x_2\,)}\,.$$

Let's check the properties given above:

(i) **Reflexivity.** For all $(x_1,\, y_1) \in \mathbb{R}^2$, it is true that

$$\boxed{(\, y_1 - y_1\,) \;=\; 3(\, x_1 - x_1\,)}\,,$$

so that $(x_1,\, y_1)\, \boldsymbol{\Re}\, (x_1,\, y_1)$ and therefore, $\Re$ is reflexive.

(ii) **Symmetry.** For all $(x_1,\, y_1),\ (x_2,\, y_2) \in \mathbb{R}^2$, it is true that

$$\boxed{(\, y_1 - y_2\,) \;=\; 3(\, x_1 - x_2\,)} \implies \boxed{(\, y_2 - y_1\,) \;=\; 3(\, x_2 - x_1\,)}\,,$$

so that $(x_1,\, y_1)\, \boldsymbol{\Re}\, (x_2,\, y_2) \implies (x_2,\, y_2)\, \boldsymbol{\Re}\, (x_1,\, y_1)$ and therefore, $\Re$ is symmetric.

(iii) **Transitivity.** For all $(x_1,\, y_1),\ (x_2,\, y_2),\ (x_3,\, y_3) \in \mathbb{R}^2$, it is true that

$$\boxed{\begin{array}{c}(\, y_1 - y_2\,) \;=\; 3(\, x_1 - x_2\,)\\ \text{and}\\ (\, y_2 - y_3\,) \;=\; 3(\, x_2 - x_3\,)\end{array}} \implies \boxed{(\, y_1 - y_3\,) \;=\; 3(\, x_1 - x_3\,)}\,,$$

because, if the two conditions on the left hold, then

$$\begin{array}{rcl}(\, y_1 - y_3\,) & = & (\, y_1 - y_2\,) + (\, y_2 - y_3\,)\\ & = & 3(\, x_1 - x_2\,) + 3(\, x_2 - x_3\,)\\ & = & 3(\, x_1 - x_3\,)\,.\end{array}$$

Thus, $\Re$ is transitive.

Therefore, we conclude again that this relation $\Re$ is indeed an equivalence relation because it is reflexive, symmetric and transitive.

Notation : In the special case where a relation $\Re$ is an **equivalence relation** it is usual to write

$$\boxed{a \sim b} \quad \text{instead of} \quad \boxed{a \Re b}\ .$$

We will adopt this convention from now on.

Equivalence Relations and Partitions: Recall that a **partition of a set** A is just a decomposition of A into a **disjoint union of subsets**, that is, we write

$$A = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots \qquad \text{where } A_i \cap A_j = \emptyset \text{ for all } i \neq j.$$

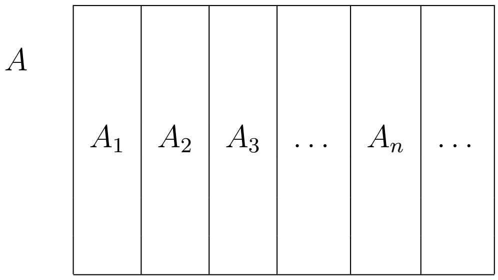

Diagram 1.

The most important thing about an equivalence relation is that

an equivalence relation on a set A
corresponds to a partition of A
and vice-versa

We will now show this, where the subsets in question are called equivalence classes, and are defined as follows:

Definition (Equivalence Classes) : Let "$\sim$" be an equivalence relation on a set A. To each $a \in A$ we associate a subset of A, denoted by $[a]$. This subset, $[a]$, is defined by

$$[a] = \{ x \in A \mid x \sim a \}$$

and is called **the equivalence class of a**.

To show that the equivalence classes do indeed form a partition of A we will begin by proving:

Proposition : If $b \in [a]$, then $[b] = [a]$

Proof :

(Step 1) Show that $b \in [a] \;\Rightarrow\; [b] \subseteq [a]$.

To see this, observe that

$$\begin{array}{rcll} c \in [b] & \Rightarrow & c \sim b & \text{By the definition of } [b]. \\ & \Rightarrow & c \sim b \;\text{ and }\; b \sim a, & \text{Because, by assumption, } b \in [a]. \\ & \Rightarrow & c \sim a & \text{By the transitivity of } \sim. \\ & \Rightarrow & c \in [a] & \text{By the definition of } [a]. \end{array}$$

Thus, we have shown that $c \in [b] \;\Rightarrow\; c \in [a]$, or in other words, $[b] \subseteq [a]$.

Just to be clear, what we have shown so far (for an equivalence relation) is that

$$\boxed{\text{cow} \in [\,\text{elephant}\,]} \;\Longrightarrow\; \boxed{[\,\text{cow}\,] \subseteq [\,\text{elephant}\,]},$$

no matter what **cow** or **elephant** might be.

(Step 2) Show that $[a] \subseteq [b]$.

We will be done, by Step 1, if we show that $a \in [b]$. To see that $a \in [b]$ is indeed the case, we go back to our starting point where we're given that $b \in [a]$. Accordingly,

$$\begin{array}{rcll} b \in [a] & \Rightarrow & b \sim a & \text{By the definition of } [a]. \\ & \Rightarrow & a \sim b & \text{By the symmetry of } \sim. \\ & \Rightarrow & a \in [b] & \text{By the definition of } [b]. \end{array}$$

Thus, we have shown that here $a \in [b]$ and, therefore by Step 1, $[a] \subseteq [b]$.

Remark : The key thing that we needed in this proof was the the fact that the relation "$\sim$" is both **symmetric and transitive**. In this proof we did not use the reflexivity of "$\sim$".

We are now ready to prove:

Theorem : If "$\sim$" is an equivalence relation on a set A, then the equivalence classes of "$\sim$" form a partition of A.

Proof :

(Step 1) Show that $A = \bigcup$ (all the equivalence classes).

This follows immediately from the **reflexivity of "$\sim$"**, because for each $a \in A$ we have

$$a \sim a \Rightarrow a \in [a].$$

That is, every element of A is in the equivalence class of itself.

(Step 2) Show that **distinct equivalence classes are disjoint**.

We show this by showing that if two equivalence classes, $[a]$ and $[b]$ say, are not disjoint, then they must be identical. So suppose that there exists $c \in [a] \cap [b]$, then:

$$c \in [a] \quad \Rightarrow \quad [c] = [a] \quad \text{By the previous Proposition.}$$

and

$$c \in [b] \quad \Rightarrow \quad [c] = [b] \quad \text{By the previous Proposition.}$$

That is, the existence of $c \in [a] \cap [b] \implies [a] = [c] = [b]$.

Remark : The converse of the Theorem above is also true. That is, if we're given a partition

$$A = A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots$$

of a set A and we define a relation $\Re$ on A, by $a \Re b \iff a,\, b \in \text{the same } A_j$ for some $j \in \mathbb{N}$, then $\Re$ is an equivalence relation (**you check this**) and the equivalence classes are the subsets $A_1, A_2, A_3, \ldots, A_n, \ldots$ Note, the union here need not be countable - but you can disregard this point if you wish.

Note : A description of the equivalence classes that we obtain from Example [1] to Example [3], given above (together with other examples) will be given in class.