

1 Tangents

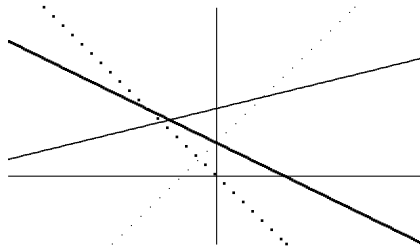
Slope

The graph of the function

$$f(x) = mx + c$$

is a straight line.

The number m is called the **slope** and it tells us which way and by how much the line is tilted, i.e. it tells us whether y is **increasing or decreasing** as x increases and how rapidly this occurs.



$m = \frac{1}{2}$, y increases as x increases;

$m = 2$, y increases more rapidly as x increases;

$m = -1$, y decreases as x increases;

$m = -2$, y decreases more rapidly as x increases.

2.1 Example

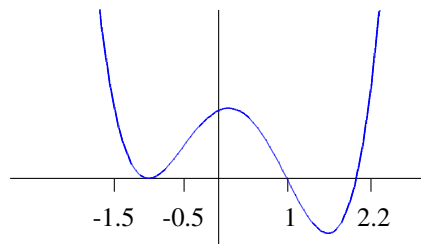


Figure 1: Graph of $f(x) = (x + 1)^2(x - 1)(x - 2)$.

When $x = -0.5$, y is increasing

When $x = 2.2$, y is increasing more rapidly;

When $x = 1$, y is decreasing;

When $x = -1.5$, y is decreasing more rapidly .

The major difference here is that the “ slope” may change from point to point.

Tangents

The **tangent** to a curve C at the point P on the curve is a straight line which touches C at P and which stays on one side of the curve near P .

The tangent touches the curve at P and may cross the curve at P but otherwise does not cross the curve near P .

The slope of the curve at $x = c$ is equal to the slope of the tangent line T at $x = c$.

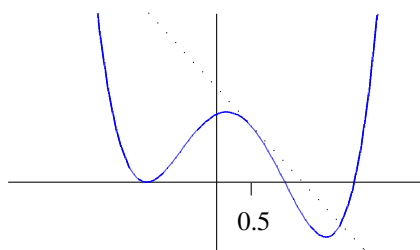


Figure 2: Tangent to $f(x) = (x + 1)^2(x - 1)(x - 2)$ at $x = 0.5$.

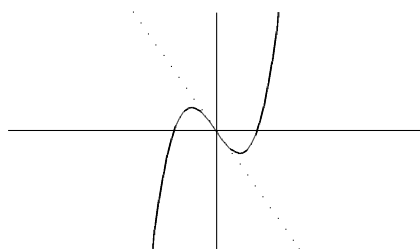


Figure 3: Tangent to $f(x) = x^3 - x$ at $x = 0$.

2.2 How to find tangents

We want to find the tangent to $y = f(x) = x^2$ at $P = (1, f(1)) = (1, 1)$

Take a point Q on the curve near P : i.e. $Q = (0.5, 0.25)$

Find the slope of the line joining P to Q :

$$\frac{1 - 0.25}{1 - 0.5} = \frac{0.75}{0.5} = 1.5.$$

The line segment PQ is called a **secant**.

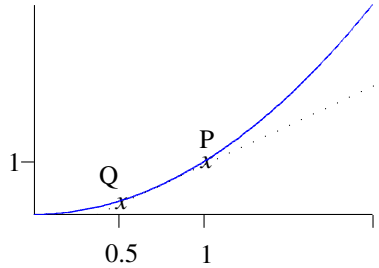


Figure 4: $f(x) = x^2$.

Let Q get closer to P .

The secant tips over and the limit of the slopes will give the slope of the tangent:

$$\text{slope of tangent} = \lim_{Q \rightarrow P} \text{slope of } P - Q.$$

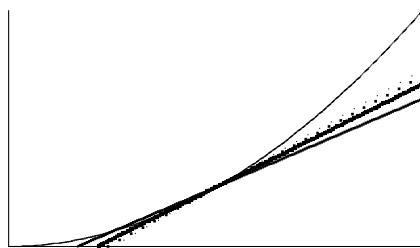


Figure 5: $f(x) = x^2$.

We can also let Q approach P from the other direction and find the limit of the slopes:

$$\text{slope of tangent} = \lim_{Q \rightarrow P} \text{slope of } P - Q.$$

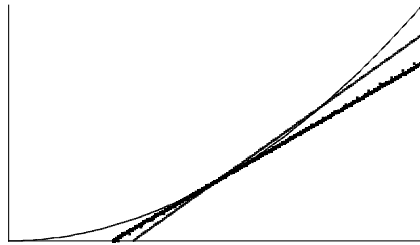


Figure 6: $f(x) = x^2$.

If Q is close to P , then $Q = (1 + h, (1 + h)^2) = (1 + h, 1 + 2h + h^2)$ where h is small.

If $h > 0$, then Q lies above and to the right of P .

If $h < 0$, then Q lies below and to the left of P .

As Q approaches P , $h \rightarrow 0$.

The slope of the secant PQ is

$$\frac{1 + 2h + h^2 - 1}{1 + h - 1} = \frac{2h + h^2}{h} = 2 + h.$$

Therefore

$$\lim_{Q \rightarrow P} \text{slope of } P - Q = \lim_{h \rightarrow 0} (2 + h) = 2.$$

The slope of the tangent to $y = x^2$ at $x = 1$ is 2.

The tangent passes through the point $P = (1, 1)$.

The **point-slope** equation of the tangent is

$$y = 1 + 2(x - 1) = 2x - 1.$$

2.3 Slope and Tangent lines

The **slope of the curve** $y = f(x)$ at the point $P = (x_0, f(x_0))$ is the number

$$m = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

The **tangent line** to the curve at P is the line through P with this slope.

2.4 Example

(i) Find the tangent to the curve $y = x^2$ when $x = 3$.

Find the Slope: The slope of the tangent to $y = x^2$ at $x = 3$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h \rightarrow 0} (6 + h) = 6. \end{aligned}$$

The tangent passes through the point $P = (3, 9)$.

The **point-slope** equation of the tangent is

$$y = 9 + 6(x - 3) = 6x - 9.$$

(iii) Find the tangent to the curve $y = \frac{1}{x}$ when $x = 2$.

Find the Slope: The slope of the tangent to $y = \frac{1}{x}$ at $x = 2$ is

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{-h}{2(2+h)}}{h} \\ &= \lim_{h \rightarrow 0} \left(-\frac{1}{2(2+h)} \right) = -\frac{1}{4}. \end{aligned}$$

The tangent passes through the point $P = \left(2, \frac{1}{2}\right)$.

The **point-slope** equation of the tangent is

$$y = \frac{1}{2} - \frac{1}{4}(x - 2) = -\frac{1}{4}x + 1.$$