1 The Binomial Distribution

The Binomial Distribution
Consider an experiment in which there are only two possible outcomes which are mutually exclusive (e.g. tossing a coin, testing a light bulb to see if it works, doing the Lotto). Let's call one of these outcomes "success" and the other one "failure". Let

$$p = \text{ probability of success}, q = \text{ probability of failure.}$$

Then

$$q = 1 - p.$$

Examples

Probability of 6 tails when we toss the coin 15 times.

Probability of getting 5 faulty light bulbs in a batch of 200.

Probability of winning the Lotto if we do it twice a week for 5 years.

Example

Suppose we throw a dice and "success" (S) is getting a a six (so "failure" (F) is getting any other number). Then

$$p = \frac{1}{6}, \quad q = \frac{5}{6}.$$

What is the probability of getting a six twice if we throw the dice three times?

The possible outcomes are

$$(FFF), (FFS), (FSF), (FSS), (SFF), (SFS), (SSF), (SSS).$$

This is not a sample space of equally likely outcomes:

The probability of (FFF) occurring is $q^3 = \frac{5^3}{6^3}$.

The probability of (SFS) occurring is $qp^2 = \frac{5}{6^3}$.

The "good" outcomes all occur with probability

$$qp^2 = \frac{5}{6^3}.$$

Look at where the S occur in these "good" outcomes:

$$(FSS) \mapsto (23), (SFS) \mapsto (13), (SSF) \mapsto (12).$$

So the number of "good" outcomes is equal to the number of unranked ways of choosing 2 numbers from 3:

$$C_2^3 = \frac{3!}{2!} = 3.$$

So the probability of getting a six twice if we throw the dice three times is

$$3 \times \frac{5}{6^3}.$$

Binomial Distribution

In each case we carry out a number of repeated and independent trials (say n trials) of the experiment and ask for the probability of getting a certain number (say k) of successes.

So we are looking at the random variable

$$X = \text{ number of successes in } n \text{ trials.}$$

The probability distribution for this random variable is called the binomial distribution and is given by

$$\begin{aligned} p(k) = P(X = k) &= \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} \\ &= \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}. \end{aligned}$$

We will write this as $B(k; n, p)$.

Example
A fair coin is tossed 6 times. Calculate

Probability that exactly 2 tails occur.

Probability that at least 4 tails occur.

Probability that no tails occur.

Probability that at least 1 head occurs.

Does this distribution satisfy the definition of a probability distribution?

$$p(k) \geq 0, \qquad \sum p(k) = 1.$$

Characteristics of a binomial random variable.
A binomial random variable is one whose probability distribution is the binomial distribution $B(k; n, p)$. We can show that

$E(X) = np$.

$V(X) = np(1-p)$.

2 The Poisson Distribution

The Poisson Distribution
In a crowd of $4,326$ at a football match, the probability that a given person will see themselves on t.v. waving at the cameras is 3%. What is the probability that 12 people will see themselves on t.v. waving at the cameras?
This is a straight application of the binomial distribution with $k = 12, n = 4326, p = 0.03$; the answer is

$$B(12; 4326, 0.03) = \begin{pmatrix} 4326 \\ 12 \end{pmatrix} (0.03)^{12} (0.97)^{4314}.$$

The large (and small) numbers here make the result very difficult to compute accurately. To get around this, we use a different distribution which has better computational properties and which gives a good approximation to $B(k; n, p)$. The problem above is that n is large and p is small. So consider the situation where n is increasing to infinity and p is decreasing to 0 in such a way that the number

$$\lambda = np$$

remains constant. We can show that in this limit we have

$$B(k; n, p) \to \frac{\lambda^k e^{-\lambda}}{k!}.$$

Poisson distribution
This is the *Poisson distribution*:

$$P(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.$$

This is the probability of getting k successes when the number of trials $n \to \infty$ and the probability $p \to 0$ in such a way that $\lambda = np$ remains constant.
This is used to *approximate* $B(k;n,p)$ in the case when n is large and p is small. In the problem above, we get ($\lambda = 4326 \times 0.03 = 129.78$)

$$B(12, 42326, 0.03) \sim P(12, 129.78) = \frac{(129.78)^{12} e^{-129.78}}{12!}.$$

Example

The probability of a defect in a mile of steel wire is 1%. A steel cable consists of 100 strands of wire, and will support its load with 99 good strands. What is the probability that a mile of cable will support its load?

Note: We need to check that the two conditions for a discrete probability distribution are satisfied.

Characteristics of a Poisson random variable.
A Poisson random variable is one whose probability distribution is the binomial distribution $B(k;n,p)$. We can show that

$E(X) = \lambda$.

$V(X) = \lambda$.

Applications

In general, $P(k;\lambda), \lambda = np$ is

a good approximation to $B(k;n,p)$ if $n \geq 20$ and $p \leq 0.05$;
an excellent approximation to $B(k;n,p)$ if $n \geq 100$ and $p \leq 0.1$.

As well as being used to approximate the binomial distribution, the Poisson distribution applies to *Poisson processes*. This is any process with the following characteristics:

The probability of getting a success in a time interval of length h depends only on h.

The probability of getting two successes in a time interval of length h is zero, provided h is sufficiently small.

The system has no memory.

Examples are radioactive decay, catching a car speeding on a motorway, a politician tells a lie, ...