1 Preliminary - How to Count

How many different ways of filling out a Lotto ticket are there?

How many different ways can you deal a full house in poker?

How many different ways can you pick a football team from a squad of 22 players?

Ranked and unranked committees

How many different ways can we choose a committee consisting of a chair, a secretary and a treasurer from a group of five people?

Adi, Billy, Ciara, David and Enda

Suppose we choose the chair first- we have 5 people to choose from.

Next choose the secretary, we have 4 choices (to go with each choice of chair)

-so far we have $5 \times 4 = 20$ options.

Lastly pick the treasurer-for each of the selections above we have 3 further options

-altogether we have

$$5 \times 4 \times 3 = 60$$

ways of choosing this ranked committee.

Unranked Committee

If the ranks are removed, how many different committees are there? That is, how many different ways can we choose a committee of 3 people from a group of 5 people?

The choice (Adi, Ciara, David) is the same as the choice (David, Ciara, Adi)

To answer this, put all the *ranked* committees which have the same set of people in them into one group. Do this for all possible sets of people. Want to know how many different sets there are. Call this number q for convenience.

group 1	group 2	group 3	...	group q

How many different ranked committees in each group?

Each group consists of a set of 3 people arranged into ranked committees of 3 people. So there are $6 = 3 \times 2 = 3!$ different ranked committees in each group.

In total, there are $60 = 5 \times 4 \times 3$ ranked committees, so we have

$$q = 10 = \frac{60}{6} = \frac{5 \times 4 \times 3}{3!} = \frac{5!}{2!3!}.$$

Ranked Committees

How many different ways can we choose a *ranked* committee of k people from a group of n people? (Must have $n \geq k$.)

$$P_k^n = n(n-1)(n-2)\cdots(n-k+2)(n-k+1).$$

Notice that there are

$$\begin{aligned} P_n^n &= n(n-1)(n-2)\cdots(n-n+2)(n-n+1) \\ &= n(n-1)(n-2)\cdots 2 \cdot 1 = n! \end{aligned}$$

ways of choosing a ranked committee of size n from n people.

Unranked Committees

If the ranks are removed, how many different committees are there?

That is, how many different ways are there of choosing k objects if there are n to choose from, given that order does not matter?

To answer this, put all the *ranked* committees which have the same set of people in them into one group. Do this for all possible sets of people. Want to know how many different sets there are. Call this number q for convenience.

group 1	group 2	group 3	...	group q

How many different ranked committees in each group? Each group consists of a set of k people arranged into ranked committees of k people.

So there are P_k^k different ranked committees in each group.

In total, there are P_k^n ranked committees, so we have

$$qP_k^k = P_k^n$$

or

$$q = \frac{P_k^n}{P_k^k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

Summary

The number of different ways of choosing k objects from a set of n, when order does not matter, is

$$C_k^n = \left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

Note that in order to use this, we will need the definition $0! = 1$.

$$\begin{aligned} \left(\begin{array}{c} n \\ 0 \end{array} \right) &= \left(\begin{array}{c} n \\ n \end{array} \right) = 1 \\ \left(\begin{array}{c} n \\ 1 \end{array} \right) &= \left(\begin{array}{c} n \\ n-1 \end{array} \right) = n \\ \left(\begin{array}{c} n \\ k \end{array} \right) &= \left(\begin{array}{c} n \\ n-k \end{array} \right) \end{aligned}$$

Examples

How many different ways can 3 items be chosen from 5? Ditto for 1,2,4,5 items.

How many different Lotto combinations are there?

How many different teams of 11 from a squad of 22?