AntiDerivatives

3.1 Antiderivative

An antiderivative of a function f is a function whose derivative is f. So F is an antiderivative of f if

$$F' = f.$$

Example:

Let $f(x) = 2x$.

Then $F(x) = x^2$ is an antiderivative of f.

However $G(x) = x^2 + 1$ is also an antiderivative as is $H(x) = x^2 - 5$.

Note F, G and H all differ by constants: $G(x) = F(x) + 1$.

3.3 Facts

If $F'(x) = 0$, for all x in an interval I, then F is constant on the interval I.

If $F'(x) = G'(x)$, for all x in an interval I, then $G(x) = F(x) + C$ on the interval I, for some constant C.

Conversely if $G(x) = F(x) + C$ on the interval I, for some constant C, then $F'(x) = G'(x)$, for all $x \in I$.

3.4 Definition

A function F is an *antiderivative* of a function f if

$$F'(x) = f(x),$$

for all x in the domain of f.

The set of all antiderivatives of f is the *indefinite integral* of f with respect to x, denoted by

$$\int f(x)dx.$$

The symbol $\int$ is an *integral sign*, the function f is the *integrand* of the integral and x is the *variable of integration*.

Once we have found one antiderivative F of f all other antiderivatives differ from F by a constant, and any function differing from F by a constant is also an antiderivative of f. We indicate this by writing

$$\int f(x)dx = F(x) + C.$$

The process of obtaining antiderivatives is called integration.

We say that f is an *integrable* function of x if we can integrate f w.r.t. x.

Rules for integrals

3.5 Rules for Integrals

(i) If $f(x)$ is a constant function, say $f(x) = a$, then

$$\int f(x)dx = \int a dx = ax + C.$$

This is true as ax is an antiderivative of a:

$$\frac{d}{dx}(ax) = a.$$

So for example

$$\int 3dx = 3x + C.$$

Power rule

(ii) If $f(x) = x^r$, where $r \neq 1$ is a rational number, then

$$\int f(x)dx = \int x^r = \frac{x^{r+1}}{r+1} + C.$$

We already know that this is true for $r = 0$. For any $r \neq -1$:

$$\frac{d}{dx}(x^{r+1}) = (r+1)x^r.$$

Hence

$$\frac{x^{r+1}}{r+1}$$

is an antiderivative of x^r.

We can apply this rule to get the following:

$$f(x) = x^7 \implies \int f(x)dx = \int x^7 dx = \frac{x^8}{8} + C;$$

$$g(x) = x^5 \implies \int g(x)dx = \int x^5 = \frac{x^6}{6} + C;$$

$$h(x) = x^{128} \implies \int h(x)dx = \int x^{128} = \frac{x^{129}}{129} + C.$$

Constant Multiple Rule

(iii) If a is any constant and f is an integrable function of x, then

$$\int af(x)dx = a\int f(x)dx.$$

Suppose that F is an antiderivative of f. Then $F'(x) = f(x)$ and

$$(aF)'(x) = aF'(x) = af(x).$$

So that aF is an antiderivative of af.

Sum and Difference Rules

(iv) If f and g are integrable functions of x, then their sum $f+g$ is integrable and

$$\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx.$$

(v) Combining (iii) and (iv) we get the difference rule.

If f and g are integrable functions of x, then their difference $f-g$ is integrable and

$$\int (f-g)(x)dx = \int f(x)dx - \int g(x)dx.$$

3.6 Integral of a Polynomial

Putting these rules together, and as we saw in the examples: If

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, then

$$\int f(x)dx = \frac{a_n x^{n+1}}{n+1} + \cdots + \frac{a_1 x^2}{2} + a_0 x + C.$$

$f(x) = 5x^6 - 2x^5 + 3x^4 - 1,$
$\int f(x)dx = \frac{5}{7}x^7 - \frac{1}{3}x^6 + \frac{3}{5}x^5 - x + C;$

$g(x) = 9x^5 - 5x^2 + 14x,$
$\int g(x)dx = \frac{3}{2}x^6 - \frac{5}{3}x^3 + 7x^2 + C;$

$h(x) = 7x^{100} - 3x^{50},$
$\int h(x)dx = \frac{7}{101}x^{101} - \frac{3}{51}x^{51} + C.$

3.7 Integration using Substitution

From the Chain rule we know that

$$\frac{d}{dx}(g(f(x))) = f'(x)g'(f(x)).$$

Therefore $g(f(x))$ is an antiderivative of $f'(x)g'(f(x))$. Hence

$$\int f'(x)g'(f(x))dx = f'(x)g'(f(x)).$$

If $u = f(x)$, then we make a substitution and use the fact that $u' = f'(x)$ and we write

$$\int f'(x)g'(f(x))dx = \int g'(u)du = f'(x)g'(f(x)).$$

In the particular case when $g(x) = x^n$, for some rational number $n \neq -1$ we have

$$\int f'(x)(f(x))^n dx = \int u'u^n dx = \int u^n du = \frac{1}{n+1}u^{n+1} + C.$$

3.8 Examples

(i) Find

$$\int (x-2)^{23}dx.$$

Here $u = x - 2,\ n = 23.$

So $u' = 1.$

Therefore

$$\begin{aligned} \int (x-2)^{23}dx &= \int u'u^{23}dx = \int u^{23}du \\ &= \frac{u^{24}}{24} + C = \frac{1}{24}(x-2)^{24} + C. \end{aligned}$$

(ii) Find

$$\int (4x^3+5)(x^4+5x-2)^3 dx.$$

Here $u = x^4 + 5x - 2,\ n = 3.$

Also $u' = 4x^3 + 5$.

Therefore

$$\begin{aligned}\textstyle\int(4x^3+5)(x^4+5x-2)^3dx &= \textstyle\int u'u^3dx = \int u^3du \\ &= \frac{u^4}{4} + C \\ &= \frac{1}{4}(x^4+5x-2)^4 + C.\end{aligned}$$

(iii) Find

$$\int \frac{7x^6+3x^2-3}{(x^7+x^3-3x+4)^5}dx.$$

Here $u = x^7 + x^3 - 3x + 4,\ n = -5$.

So $u' = 7x^6 + 3x^2 - 3$.

Therefore

$$\begin{aligned}\int \frac{7x^6+3x^2-3}{(x^7+x^3-3x+4)^5}dx &= \int u'u^{-5}dx = \int u^{-5}du \\ &= -\frac{u^{-4}}{4} + C\end{aligned}$$

$$= -\frac{1}{4(x^7+x^3-3x+4)^4} + C.$$

(iv) Find

$$\int \frac{x+1}{(x^2+2x-7)^8}dx.$$

Here $u = x^2 + 2x - 7,\ n = -8$.

So $u' = 2x + 2$.

Therefore

$$\begin{aligned}\int \frac{x+1}{(x^2+2x-7)^8}dx &= \frac{1}{2}\int \frac{2x+2}{(x^2+2x-7)^8}dx \\ &= \frac{1}{2}\int u' u^{-8} dx \\ &= \frac{1}{2}\int u^{-8} du \\ &= -\frac{u^{-7}}{14} + C = -\frac{1}{14}(x^2+2x-7)^{-7} + C.\end{aligned}$$

3.9 Integration by Parts

We know from the product rule that if f and g are differentiable functions of x, then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x).$$

Therefore $f(x)g(x)$ is an antiderivative of $f(x)g'(x) + f'(x)g(x)$. Hence

$$\begin{aligned}\textstyle\int f(x)g'(x)dx + \int f'(x)g(x)dx &= \textstyle\int f(x)g'(x) + f'(x)g(x)dx \\ &= f(x)g(x).\end{aligned}$$

Rearranging this we get

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

We can also write this as

$$\int uv'dx = uv - \int u'vdx.$$

3.10 Example

(i) Find

$$\int x(x+2)^{19}dx.$$

Set $u = x \Longrightarrow u' = 1$.

Set $v' = (x+2)^{19} \Longrightarrow v = \frac{1}{20}(x+2)^{20}$.

Therefore

$$\int x(x+2)^{19}dx = \frac{x}{20}(x+2)^{20} - \int \frac{1}{20}(x+2)^{20}dx$$
$$= \frac{x}{20}(x+2)^{20} - \frac{1}{420}(x+2)^{21} + C.$$

(ii) Find
$$\int x^2(x+2)^{18}dx.$$

Set $u = x^2 \Longrightarrow u' = 2x$.

Set $v' = (x+2)^{18} \Longrightarrow v = \frac{1}{19}(x+2)^{19}$.

Therefore

$$\int x^2(x+2)^{18}dx = \frac{x^2}{19}(x+2)^{19} - \frac{2}{19}\int x(x+2)^{19}dx$$
$$= \frac{x^2}{19}(x+2)^{19} - \frac{x}{190}(x+2)^{20}$$
$$+\frac{1}{3990}(x+2)^{21} + C.$$

The Definite Integral

We can work out the area under the line $y = x$ between $x = a$ and $x = b$ (as done on the board. However for other curves it is not so easy and we can only make estimates.

3.11 Example: Let $f(x) = x^2$. What is the area under the curve between $x = 0$ and $x = 1$?

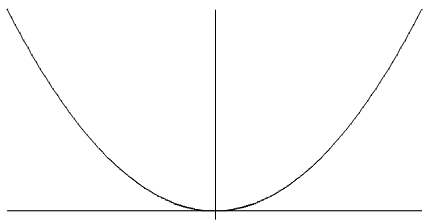

Figure 1: Graph of $f(x) = x^2$.

We can make an approximation of the area by measuring the area of rectangles which have base a subinterval of $[0, 1]$ and height x^2, for some x (as detailed below).

Divide the interval $[0, 1]$ into four subintervals

$$[0, 0.25],\ [0.25, 0.5],\ [0.5, 0.75],\ [0.75, 1].$$

Place rectangles with bases on these subintervals and heights

$$0,\ 0.25^2 = 0.0625,\ 0.5^2 = 0.25,\ 0.75^2 = 0.5625$$

respectively.

The diagram (on board) shows that these rectangles touch the curve but are contained beneath the curve. The sum of the areas of these rectangles underestimates the area under the curve.

The area of the four rectangles is

$$0 + 0.25 \times 0.0625 + 0.25 \times 0.25 + 0.25 \times 0.5625 = 0.21875.$$

We can also approximate the area by placing rectangles on these subintervals and heights

$$0.25^2 = 0.0625,\ 0.5^2 = 0.25,\ 0.75^2 = 0.5625,\ 1$$

respectively.

The diagram (on board) shows that these rectangles touch the curve but the sum of the areas of these rectangles overestimates the area under the curve.

The area of the four rectangles is

$$0.25 \times 0.0625 + 0.25 \times 0.25 + 0.25 \times 0.5625 + 0.25 = 0.46875.$$

We can make a better approximation of the area by increasing the number of rectangles.

Divide the interval $[0, 1]$ into ten subintervals

$$[0, 0.1],\ [0.1, 0.2],\ \ldots,\ [0.9, 1].$$

Place rectangles with bases on these subintervals and heights

$$0,\ 0.1^2 = 0.01,\ 0.2^2 = 0.04, \ldots, 0.9^2 = 0.81$$

respectively.

The diagram (on board) shows that these rectangles touch the curve but are contained beneath the curve. The sum of the areas of these rectangles underestimates the area under the curve.

The area of the ten rectangles is

$$0 + 0.1 \times 0.01 + 0.1 \times 0.04 + \cdots + 0.1 \times 0.81 = 0.285.$$

We can also approximate the area by placing rectangles on these subintervals and heights

$$0.1^2 = 0.01,\ 0.2^2 = 0.04,\ \ldots,\ 1$$

respectively. The diagram (on board) shows that these rectangles touch the curve but the sum of the areas of these rectangles overestimates the area under the curve.

The area of the ten rectangles is

$$0.1 \times 0.01 + 0.1 \times 0.04 + \cdots + 0.1 \times 1 = 0.385.$$

Divide the interval $[0, 1]$ into n subintervals of equal length

$$\left[0, \frac{1}{n}\right],\ \left[\frac{1}{n}, \frac{2}{n}\right],\ \ldots,\ \left[\frac{n-1}{n}, 1\right].$$

Place rectangles with bases on these subintervals and heights

$$0,\ \frac{1}{n^2}, \frac{4}{n^2}, \ldots, \frac{(n-1)^2}{n^2}$$

respectively. The sum of the areas of these rectangles underestimates the area under the curve.

The area of the n rectangles is

$$\begin{aligned} \frac{1}{n}\left(0 + \frac{1}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right) \quad &= \frac{1}{n}\sum_{i=1}^{n-1}\frac{i^2}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n-1} i^2 \\ &= \frac{(n-1)n(2n-1)}{6n^3}. \end{aligned}$$

We can also approximate the area by placing rectangles on these subintervals and heights

$$\frac{1}{n^2}, \frac{4}{n^2}, \ldots, 1$$

respectively.

Again these rectangles touch the curve but the sum of the areas of these rectangles overestimates the area under the curve.

The area of the n rectangles is

$$\begin{aligned} \frac{1}{n}\left(\frac{1}{n^2} + \cdots + \frac{(n-1)^2}{n^2} + 1\right) &= \frac{1}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

$$\lim_{n\to\infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3}.$$

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}.$$

This limit is called the ***definite integral***

$$\int_0^1 x^2 dx.$$

Continuous functions

A function f is said to be continuous on the interval $[a, b]$ if

$$\lim_{x\to c} f(x) = f(c),$$

for all $c \in [a, b]$.

Note polynomials and rational functions are continuous on any interval on which they are defined.

3.12 Riemann Integrals

A *partition* P of the interval $[a, b]$ is any finite set of points $t_0, t_1, \ldots, t_n$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Quite often we will consider partitions $P_n = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with equal cells i.e.

$$t_i - t_{i-1} = \frac{(b-a)}{n}.$$

In this case

$$t_0 = a,\ t_1 = a + \frac{(b-a)}{n},\ t_2 = a + \frac{2(b-a)}{n}, \ldots,$$

$$\ldots, t_i = a + \tfrac{i(b-a)}{n}, \ldots, t_n = b.$$

Set $\Delta_i = t_i - t_{i-1}$.

If we are considering a partition with equal cells, then $\Delta_i = \dfrac{1}{n}$, for all $1 \leq i \leq n$.

Consider a function f which is continuous on the interval $[a, b]$.

Choose an arbitrary t_i^* in the subinterval $[t_{i-1}, t_i]$ and evaluate $f(t_i^*)$.

If $f \geq 0$ on $[a, b]$, then $f(t_i^*)\Delta_i$ is the area of a rectangle with base Δ_i and height $f(t_i^*)$. Furthermore if n is large, then

$$\sum_{i=1}^{n} f(t_i^*)\Delta_i$$

is a good estimate of the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

Given any partition P set $\Delta_P = \max_i \Delta_i$.

Let $\mathcal{P}$ denote the set of all partitions of the interval $[a, b]$.

If

$$\lim_{P \in \mathcal{P},\, \Delta_P \to 0} \sum_{i=1}^{n} f(t_i^*)\Delta_i$$

exists, then f is said to be integrable on $[a, b]$ and the limit is called the *definite integral*

$$\int_a^b f(x)dx.$$

In fact if this limit exists we can just consider the partitions P_n so $\Delta_{P_n} = \frac{1}{n}$.

If $f \geq 0$, on $[a, b]$, then $\int_a^b f(x)dx$ is equal to the area under the curve $y = f(x)$ between $x = a$ and $x = b$.

3.13 Example

Consider $f(x) = x^2$ on the interval $[a, b]$ where $0 \leq a \leq b$.

Divide the interval $[a, b]$ into n subintervals of equal length

$$\left[a, a + \frac{b-a}{n}\right], \left[a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}\right], \ldots$$
$$\ldots, \left[a + \frac{(n-1)(b-a)}{n}, b\right].$$

Place rectangles with bases on these subintervals and heights

$$a^2, \left(a + \frac{b-a}{n}\right)^2, , \ldots, \left(a + \frac{(n-1)(b-a)}{n}\right)^2$$

respectively.

The area of the n rectangles is

$$\frac{b-a}{n} \sum_{i=1}^{n-1} \left(a + \frac{i(b-a)}{n}\right)^2$$
$$= \frac{b-a}{n} \sum_{i=0}^{n-1} \left(a^2 + \frac{2ai(b-a)}{n} + \frac{i^2(b-a)^2}{n^2}\right)$$
$$= \frac{b-a}{n} \sum_{i=0}^{n-1} a^2 + \frac{b-a}{n} \sum_{i=0}^{n-1} \frac{2ai(b-a)}{n} + \frac{b-a}{n} \sum_{i=0}^{n-1} \frac{i^2(b-a)^2}{n^2}$$
$$= (b-a)a^2 + \frac{2a(b-a)^2}{n^2} \sum_{i=0}^{n-1} i + \frac{(b-a)^3}{n^3} \sum_{i=0}^{n-1} i^2$$
$$= \frac{(n-1)(b-a)a^2}{n} + \frac{a(b-a)^2(n-1)}{n}$$
$$+ \frac{(b-a)^3(n-1)(2n-1)}{6n^2}.$$

As $n \to \infty$ the area approximated is

$$
\begin{aligned}
&(b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} \\
&= ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{b^3}{3} - b^2a + a^2b - \frac{a^3}{3} \\
&= \frac{b^3}{3} - \frac{a^3}{3}.
\end{aligned}
$$

We can also approximate the area by placing rectangles on these subintervals and heights

$$\left(a + \frac{b-a}{n}\right)^2, \ldots, \left(a + \frac{(n-1)(b-a)}{n}\right)^2, b^2$$

respectively. The area of the n rectangles is

$$
\begin{aligned}
&\frac{b-a}{n} \sum_{i=1}^{n} \left(a + \frac{i(b-a)}{n}\right)^2 \\
&= (b-a)a^2 + \frac{a(b-a)^2(n+1)}{n} \\
&+ \frac{(b-a)^3(n+1)(2n+1)}{6n^2}.
\end{aligned}
$$

As $n \to \infty$ the area approximated is

$$(b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3}{3} - \frac{a^3}{3}.$$

Hence

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{b^3}.$$

3.14 Average/Mean Value

If f is integrable on $[a, b]$, then its *average value* on $[a, b]$ is

$$av(f) = \frac{1}{b-a} \int_a^b f(x) dx.$$

We make an estimate of the average value by taking a partition $P_n = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with cells of equal length $\Delta_i = \frac{(b-a)}{n}$.

Evaluate f at some $t_i^* \in [t_{i-1}, t_i]$. The average of these n values is

$$\frac{f(t_1^*) + \cdots + f(t_n^*)}{n} = \sum_{i=1}^{n} \frac{f(t_i^*)}{n} = \frac{1}{b-a} \sum_{i=1}^{n} \Delta_i f(t_i^*).$$

As $n \to \infty$, we get a better estimate of the average value of f on $[a, b] : av(f) = \lim_{n\to\infty} \frac{1}{b-a} \sum_{i=1}^{n} \Delta_i f(t_i^*) = \frac{1}{b-a} \int_a^b f(x)dx.$

3.15 Rules for definite integrals

Suppose that $a \le b \le c$, f, g are continuous on $[a, c]$ and that k is a constant.

1. Order of integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (a definition).
2. Zero: $\int_a^a f(x)dx = 0$ (a definition).
3. Constant Multiple: $\int_a^b kf(x)dx = k\int_a^b f(x)dx.$

   This is true as

   $$\lim_{P\in\mathcal{P},\, \Delta_P\to 0} \sum_{i=1}^{n} kf(t_i^*)\Delta_i = k \lim_{P\in\mathcal{P},\, \Delta_P\to 0} \sum_{i=1}^{n} f(t_i^*)\Delta_i.$$

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

   This is true as

   $\lim_{P\in\mathcal{P},\, \Delta_P\to 0} \sum_{i=1}^{n} (f(t_i^*) \pm g(t_i^*))\Delta_i$

   $= \lim_{P\in\mathcal{P},\, \Delta_P\to 0} \sum_{i=1}^{n} f(t_i^*)\Delta_i \pm \lim_{P\in\mathcal{P},\, \Delta_P\to 0} \sum_{i=1}^{n} g(t_i^*)\Delta_i.$

5. Additivity: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$
6. Max-Min Inequality: if $\max\ f$ and $\min\ f$ are the maximum and minimum values of f on $[a, b]$, then

   $$\min\ f \le av(f) = \frac{1}{b-a}\int_a^b f(x)dx \le \max\ f$$

   $$\implies \min\ f(b-a) \le \int_a^b f(x)dx \le \max\ f(b-a).$$

7. Domination: If $f(x) \geq g(x)$, for all $x \in [a,b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

If $f(t_i^*) - g(t_i^*) \geq 0$, then

$$\lim_{P\in\mathcal{P},\, \Delta_P \to 0} \sum_{i=1}^{n} (f(t_i^*) - g(t_i^*))\Delta_i \geq 0.$$

3.16 Fundamental Theorem of Calculus

Part I: If f is continuous on $[a,b]$, then the function

$$F(x) = \int_a^x f(t)\, dt$$

has a derivative at every point x in $[a,b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\, dt = f(x)$$

(see explanation on board).

3.16 Fundamental Theorem of Calculus

Part II: If f is continuous on $[a,b]$ and G is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\, dx = G(b) - G(a) = [G(x)]_a^b\,.$$

Let $F(x) = \displaystyle\int_a^x f(t)\, dt$. Then by Part I, F is also an antiderivative of f. So $F(x) = G(x) + c$, for some constant c and all $x \in [a,b]$. Note $F(a) = 0$.

$$\begin{aligned}\int_a^b f(t)dt &= F(b) = F(b) - F(a)\\ &= (G(b)+c) - (G(a)+c) = G(b) - G(a).\end{aligned}$$

3.17 Example

(i)

$$\frac{d}{dx}\int_a^x (5t^6 - t^4 + 2)\,dt = 5x^6 - x^4 + 2.$$

(ii)

$$\frac{d}{dx}\int_2^x \frac{4t-1}{t^3-4t^2+1}\,dt = \frac{4x-1}{x^3-4x^2+1}.$$

(iii)

$$\frac{d}{dx}\int_{-3}^x (t^{10}+8t^5)^{21}\,dt = (x^{10}+8x^5)^{21}.$$

(iv)

$$\begin{aligned}\int_1^2 x^3 + 2x\,dx &= \left[\frac{x^4}{4} + x^2\right]_1^2 \\ &= \left(\frac{16}{4}+4\right) - \left(\frac{1}{4}+1\right) = \frac{27}{4}.\end{aligned}$$

(v)

$$\int_{-3}^{-2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-3}^{-2} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

(vi) Find the area between the x-axis and the graph of the curve $f(x) = x^3 - x^2 - 2x$ for $-1 \leq x \leq 2$.

First find the zeros of f :

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2).$$

Therefore the zeros are $x = 0,\ -1$ and 2.

The zeros partition $[-1, 2]$ into two subintervals: $[-1, 0]$ on which $f \geq 0$ and $[0, 2]$ on which $f \leq 0$.

We integrate f over each subinterval and add the absolute values

$$\int_{-1}^{0} x^3 - x^2 - 2x\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0}$$

$$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}$$

$$\int_{0}^{2} x^3 - x^2 - 2x\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= \left(4 - \frac{8}{3} - 4\right) - 0 = -\frac{8}{3}.$$

So the total area is

$$\frac{5}{12} + \frac{8}{3} = \frac{37}{12}.$$

3.18 Substitution in Definite Integrals

The formula is as follows

$$\int_{a}^{b} g(f(x))f'(x)\,dx = \int_{f(a)}^{f(b)} g(u)\,du$$

where $u = f(x)$, $du = f'(x)dx$ and we integrate from $f(a)$ to $f(b)$.

If G is an antiderivative of g, then

$$\frac{d}{dx}G(f(x)) = G'(f(x))f'(x)$$

$$= g(f(x))g'(x).$$

3.18 Substitution in Definite Integrals

$$\int_{a}^{b} g(f(x))g'(x)\,dx = [G(f(x))]_a^b$$

$$= G(f(b)) - G(f(a))$$

$$= [G(u)]_{u=f(a)}^{u=f(b)}$$

$$= \int_{f(a)}^{f(b)} G(u)\,du.$$

In particular if $n \neq -1$ and $u = f(x)^n$, then

$$\int_a^b f(x)^n f'(x)\,dx = \int_{f(a)}^{f(b)} u^n\,du.$$

3.19 Example

(i)

$$\begin{aligned}\int_1^2 (2x-1)(x^2-x)^5\,dx &= \int_0^2 u^5\,du \\ &= \left[\frac{u^6}{6}\right]_0^2 \\ &= \frac{64}{6}.\end{aligned}$$

Here we made the substitution $u = x^2 - x$, $u(2) = 2$, $u(1) = 0$.

(ii)

$$\begin{aligned}\int_{-1}^1 \frac{5r}{(4+r^2)^2}\,dr &= \frac{5}{2}\int_{-1}^1 \frac{2r}{(4+r^2)^2}\,dr \\ &= \frac{5}{2}\int_5^5 \frac{1}{u^2}\,du \\ &= 0.\end{aligned}$$

Here we made the substitution $u = 4 + r^2$, $u(1) = 5 = u(-1)$.

(iii)

$$\begin{aligned}\int_0^1 \frac{5r}{(4+r^2)^2}\,dr &= \frac{5}{2}\int_0^1 \frac{2r}{(4+r^2)^2}\,dr \\ &= \frac{5}{2}\int_4^5 \frac{1}{u^2}\,du \\ &= \frac{5}{2}\left[-\frac{1}{u}\right]_4^5 \\ &= \frac{5}{2}\left(-\frac{1}{5}+\frac{1}{4}\right) = \frac{1}{8}.\end{aligned}$$

Here we made the substitution $u = 4 + r^2$, $u(1) = 5$, $u(0) = 4$.

3.20 Integration by parts for Definite Integrals

The formula is as follows

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

or

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx$$

3.21 Example

(i)

$$\int_0^1 x(x+1)^3\,dx$$

Set $u = x$, then $u' = 1$.

Set $v' = (x+1)^3$, then $v = \frac{1}{4}(x+1)^4$.

So

$$\begin{aligned}\int_0^1 x(x+1)^3\,dx &= \left[\frac{x}{4}(x+1)^4\right]_0^1 - \int_0^1 \frac{1}{4}(x+1)^4 dx \\ &= 4 - \left[\frac{1}{20}(x+1)^5\right]_0^1 \\ &= \frac{49}{20}.\end{aligned}$$

(ii)

$$\int_{-3}^{-1} x(x+3)^5\,dx$$

Set $u = x$, then $u' = 1$.

Set $v' = (x+3)^5$, then $v = \frac{1}{6}(x+3)^6$.

So

$$\begin{aligned}\int_{-3}^{-1} x(x+3)^5\,dx &= \left[\frac{x}{6}(x+3)^6\right]_{-3}^{-1} - \int_{-3}^{-1} \frac{1}{6}(x+3)^6 dx \\ &= -\frac{32}{3} - \left[\frac{1}{42}(x+3)^7\right]_{-3}^{-1} \\ &= -\frac{32}{3} - \frac{64}{21} = -\frac{288}{21}.\end{aligned}$$