SETS 1

The Notion of a Set: In many practical situations we wish to consider objects with certain particular properties or characteristics. For example, consider:

> All people who sat **Ordinary Level Mathematics** in the Leaving Certificate Examination this year.

When considered as a collection in this way the objects in question are said to form a set. There are certain technical reasons (to be discussed later) why **we cannot consider the set of everything**, so in any given situation we specify a **universal set** U containing all objects of current interest.

In this case, we might take the universal set to be "all people who were alive for any period of this year". Or we might take as universal set "all people who were born prior to January of this year". The most important thing is that the universal set should contain all of the objects of current interest.

Furthermore, what is absolutely essential when considering a set A, is that there should be no ambiguity about which objects are in A and which are not. For example, the phrase:

'All people in this class who have red hair'

is **not** sufficient to specify a set unless we make it more precise exactly what we mean by 'red hair'.

Henceforth, a set S will consist of a well-defined collection of objects 's' each taken from the same underlying universal set, which may or may not be mentioned explicitly.

Notation: We will write $s \in S$ or $S \ni s$ to signify that the object s is in the set S and write $s \notin S$ or $S \not\ni s$ to signify that the object s is **not** in the set S. The objects in S are called **the elements of** S. Sets will usually be described in one of the following ways:

(i) By listing: $A = \{a, e, i, o, u\}$.

(ii) By description: $A = \{n \text{ an integer} \mid n = 2m + 1, \text{with } m \text{ an integer}\}$.

The matching brackets "$\{\quad\}$" are read "**the set of all**" while the symbol "$\mid$" is read "**such that**". Sometimes "$:$" is used instead of "$\mid$". Thus the the description above is read:

$$A = \boxed{\begin{array}{c}\textbf{the set of all } n \text{ an integer } \textbf{such that} \\ n = 2m+1, \text{ where } m \text{ is an integer.}\end{array}}.$$

(iii) By performing various operations on other sets, which we now explain.

Subset: The set B is called a **subset** of the set A if every element of B is also an element of A. This is signified by writing $B \subseteq A$ or $A \supseteq B$. If there is at least one element of the set A which is not the subset B of A then we say B is a **proper subset** of A and write $B \subset A$ or $A \supset B$. In terms of a **Venn diagram** (named after John Venn a 19th century English mathematician) this is represented by:

Diagram 1.

Equality: Sets A and B are said to be **equal** if and only if $B \subseteq A$ and $A \subseteq B$. That is if and only if they consist of exactly the same elements. We write $A = B$ to signify equality of A and B.

Union: For any two sets A and B their **union** (denoted $A \cup B$) is defined by

$$A \cup B := \{x \,|\, x \in A \textbf{ or } x \in B\}$$

Note the 'or' here is the **inclusive or**, that is x is either in A or in B or in both A and B. This is represented by a Venn diagram as:

Diagram 2.

Intersection: For any two sets A and B their **intersection** (denoted $A \cap B$) is defined by

$$A \cap B := \{x \,|\, x \in A \textbf{ and } x \in B\}$$

As a Venn diagram this is:

Diagram 3.

Difference: For any two sets A and B the symbol $A \setminus B$ denotes a set, called A **difference** B, which is defined by

$$A \setminus B := \{x \in A \,|\, x \notin B\}.$$

$A \setminus B$ is sometimes called A **less** B. As a Venn diagram this is:

Diagram 4.

Complement: For any set A the set $U \setminus A$ (where U is the universal set) is called the **complement** of A and is usually denoted by $\overline{A}$. Thus $\overline{A}$ consists of all elements (in the universal set) except those in A. As a Venn diagram this is:

Diagram 5.

In particular, it is clear that $A \setminus B = A \cap \overline{B}$.

Empty Set: The **empty set** is the set containing no elements and is usually denoted by $\emptyset$ or $\{\ \}$. Note, when the universal set is a set of numbers (including 0), then the empty set is not the same as the set consisting of the number zero. That is $\{\ \} \neq \{0\}$.

Disjoint Sets: Sets A and B are said to be **disjoint** if they have no elements in common, or expressed symbolically if $A \cap B = \emptyset$.

Diagram 6.

****

EXAMPLE 1, (**Language**): In everyday life we can get along fine with a hazy notion of what we mean by a language, this lack of precision will not do in the case of computing. The following is a sufficiently general and mathematically precise description of language to incorporate most languages, be they of the everyday or of the computing variety.

We begin with an **alphabet** Σ. That is, a finite set of (distinct) symbols or objects. For example we might take Σ to be the set

{ a, b, c, ... x, y, z }

which is commonly used when writing English. Or we may prefer to take Σ to be the alphabet

{ a, á, b, ḃ, c, ċ ... u, ú }

which was commonly used when writing Irish until recently. Or indeed, we may take

$$\Sigma = \{0, 1\}.$$

Given an **alphabet** Σ we form **finite strings** of symbols from this alphabet called **words**. Using the English alphabet we might form the words:

a, aa, abb, random, rubbish, ...

Once the **alphabet** Σ is fixed, the set of all words (i.e. finite strings) which it is possible to make from Σ is denoted by Σ^*, this is an infinite set. By definition, a **language** is any subset of Σ^*.

Restriction on Σ: The elements in Σ are arbitrary except that they are not allowed to be strings of symbols from Σ. Take for instance the Irish alphabet

$$\{ \text{a, á, b, ḃ, c, ċ} \ldots \text{u, ú} \}$$

which was changed in the late 1940's early 1950's by the following rule:

> Any letter with a séiṁiú (for example ṁ) was to be deleted from the alphabet and any word where such a letter séiṁiṫe appeared was to be replaced by a new word with the letter séiṁiṫe replaced by the corresponding letter followed by an "h".

For example "ṁ" is replaced by "mh" and "ṫ" is replaced by "th" so that the word "séiṁiṫe" is replaced by the word "séimhithe".

If instead of deleting the letters $\{$ ḃ, ċ … ṫ $\}$ from Σ we decided to replace them by $\{$ bh, ch, … th $\}$ then the alphabet

$$\{ \text{a, á, b, ḃ, c, ċ} \ldots, \text{u, ú} \}$$

would be replaced by

$$\{ \text{a, á, b, bh, c, ch}, \ldots, \text{u, ú} \}.$$

This may seem to be fine but confusion would arise when translating from one alphabet to the other.

For example, how do we decide whether something like "bhh" is to mean:

"bh" followed by "h"

or

"b" followed by "h" followed by "h"

Note: When translating from the alphabet

$$\{ \text{a, á, b, ḃ, c, ċ}, \ldots \text{h}, \ldots \text{u, ú} \}$$

to the alphabet

$$\{ \text{a, á, b, c}, \ldots \text{h}, \ldots \text{u, ú} \}$$

information will be lost in general (why?). However, in the case of the Irish language this is not the case (why?).

****

RUSSELL'S PARADOX: The simple-minded (non-axiomatic) approach that we have taken to Set Theory leads to problems. One of the founders of Set Theory, Georg Cantor (1845-1918), described a set as:

any collection of definite or seperate objects of our intuition or our thought.

Using this description of a set, Bertrand Russell (1872-1970) attempted to define the following particular set $\mathcal{S}$ by the rule:

$$\mathcal{S} := \{A \,|\, A \text{ is a set and } A \notin A\}.$$

Note, if you allow that a set has (among all of its attributes) the attribute of being a thought, then the set $\mathcal{T}$ of all thoughts is a set containing itself as an element, that is $\mathcal{T} \in \mathcal{T}$. On the other hand the set $\mathcal{H}$ of all horses is not itself a horse so that $\mathcal{H} \notin \mathcal{H}$. Accordingly, some sets contain themselves as elements and some do not. Let's go back now to our proposed set $\mathcal{S}$. If this is a set (and it should be according to Cantor) we must be able to say whether or not $\mathcal{S} \in \mathcal{S}$.

For the argument that will follow let us restate the definition of the set $\mathcal{S}$. We will write $\mathcal{S}$ in RED when we want to emphasize that it is a SET as opposed to an element. Thus for any set A we have (by the definition of the set $\mathcal{S}$) that:

$$\boxed{A \in \mathcal{S}} \iff \boxed{A \notin A}\,.$$

Note: We write the set A in BLUE when we want to emphasize that it is an ELEMENT as opposed to a set. Regarding $\mathcal{S}$ we have two cases:

Case 1, We claim that $\mathcal{S} \in \mathcal{S}$.

Let's check this claim against the definition of the set $\mathcal{S}$ given above. On replacing A by $\mathcal{S}$, we find that:

$$\boxed{\mathcal{S} \in \mathcal{S}} \iff \boxed{\mathcal{S} \notin \mathcal{S}}\,.$$

But this is rubbish so that **Case 1** cannot hold.

Case 2, We claim that $\mathcal{S} \notin \mathcal{S}$.

Again, on replacing A by $\mathcal{S}$, we check this claim against the definition of the set $\mathcal{S}$ given above to find that:

$$\boxed{\mathcal{S} \notin \mathcal{S}} \iff \boxed{\mathcal{S} \in \mathcal{S}}\,.$$

Which again is rubbish so that **Case 2** cannot hold either.

These contradictions lead us to conclude that the object $\boldsymbol{\mathcal{S}}$ described above is not a set. This is Russell's Paradox. One gets out of this bind in Axiomatic Set Theory by saying that we have a **Universal Set** U so that the definition of $\boldsymbol{\mathcal{S}}$ now is:

$$\boldsymbol{\mathcal{S}} := \{A \in U \,|\, A \text{ is a set and } A \notin A\}.$$

We proceed again as as in **Case 1** above but now the contradiction leads to **either** the validity of **Case 2** **or** to the conclusion that $\boldsymbol{\mathcal{S}} \notin U$. Going on to **Case 2** the contradiction here finally leads us to the conclusion that $\boldsymbol{\mathcal{S}} \notin U$. For this reason **We Cannot Have The Universal Set Of EVERYTHING**.

****

Combining Operations: When we have more than two sets (A, B and C say) we can combine them in various ways. For example we might form the sets:

$$(A \cup B) \cap C, \quad A \cup (B \cap C), \quad (A \cap B) \cup C, \quad A \cap (B \cup C)$$

and many more combinations of this type. We will investigate presently when such combinations lead to the same set, however, just now we state the following particularly easy identities:

(i) $(A \cap B) \cap C = A \cap (B \cap C)$ and

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$.

Because of these identities, expressions like

$$A \cap B \cap C \quad \text{and} \quad A \cup B \cup C$$

make sense. That is, the order of combination (which is a binary operation) is unimportant in these two cases. Indeed, we can describe these sets as follows:

$$A \cap B \cap C = \{x \,|\, x \in \textbf{ each one} \text{ of the sets } A,\ B \textbf{ and } C\}$$

and

$$A \cup B \cup C = \{x \,|\, x \in \textbf{ at least one} \text{ of the sets } A,\ B \textbf{ or } C\}.$$

These ideas apply equally well to a collection of sets $A_1, A_2, \ldots, A_n, \ldots$

In the case of **intersections** we obtain

$$\begin{aligned}\bigcap_{n=1}^{\infty} A_n &:= A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots \\ &= \{x \,|\, x \in \textbf{ each one} \text{ of the sets } A_1, A_2, \ldots, A_n, \ldots\}\end{aligned}$$

and in the case of **unions** we have

$$\begin{aligned}\bigcup_{n=1}^{\infty} A_n &:= A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots \\ &= \{x \,|\, x \in \textbf{ at least one } \text{of the sets } A_1,\, A_2,\, \ldots,\, A_n,\, \ldots\}.\end{aligned}$$

Disjoint Union: If sets $A_1, A_2, \ldots, A_n, \ldots$ are **pair-wise disjoint**, that is if $A_i \cap A_j = \emptyset$ for all **distinct indices** i and j, then their union is called a **disjoint union**. Because this notion is of such importance in applications we sometimes emphasise that the union is disjoint by writing

$$\bigsqcup_{n=1}^{\infty} A_n \quad \text{instead of} \quad \bigcup_{n=1}^{\infty} A_n.$$

Number of Elements: If a set A contains only a **finite number** of elements we set

$$|A| := \text{the number of elements in } A$$

and if the number of elements in A is **infinite** we set $|A| := \infty$. We remark that (with a more sophisticated way of counting) there is more than one infinity. That is, some infinities are are 'bigger' than others. When counting the number of elements in a set using this more sophisticated counting technique one calls the number of elements in a set its **cardinality**.

Partitions and Counting: Many important results in Mathematics depend on counting the number of elements in a set in two different ways. All methods of counting depend on the following simple observation:

> If a (finite) set S is a **disjoint union** of sets $A_1, A_2, \ldots, A_m$, then the **number of elements in** S is given by
>
> $$|S| = |A_1| + |A_2| + \cdots + |A_m|.$$

Diagram 7.

When a set S is a **disjoint union** of sets $A_1, A_2, \ldots, A_m$ we say that the subsets $A_1, A_2, \ldots, A_m$ form a **partition of** S.