

SETS 2

In Question 1 of Tutorial 2, you are asked to draw pictures which will convince yourself of the validity of the following laws:

(i) **Idempotent Laws**

$$\begin{cases} A \cap A = A \\ A \cup A = A \end{cases}$$

(ii) **Commutative Laws**

$$\begin{cases} A \cap B = B \cap A \\ A \cup B = B \cup A \end{cases}$$

(iii) **Associative Laws**

$$\begin{cases} A \cap (B \cap C) = (A \cap B) \cap C \\ A \cup (B \cup C) = (A \cup B) \cup C \end{cases}$$

(iv) **Absorption Laws**

$$\begin{cases} A \cap (A \cup B) = A \\ A \cup (A \cap B) = A \end{cases}$$

(v) **Distributive Laws**

$$\begin{cases} A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{cases}$$

(vi) **Involution Law**

$$\overline{\overline{A}} = A$$

(vii) **De Morgan's Laws**

$$\begin{cases} \overline{(A \cap B)} = \overline{A} \cup \overline{B} \\ \overline{(A \cup B)} = \overline{A} \cap \overline{B} \end{cases}$$

(viii) **Identity Laws**

$$\begin{cases} A \cap \emptyset = \emptyset \\ A \cap U = A \\ A \cup \emptyset = A \\ A \cup U = U \end{cases}$$

(ix) **Complement Laws**

$$\begin{cases} \overline{\emptyset} = U \\ \overline{U} = \emptyset \\ A \cap \overline{A} = \emptyset \\ A \cup \overline{A} = U \end{cases}$$

Many of the statements above are obvious, for example statements (i), (ii), (iii), (iv) and (vi). However, others such as (v) and (vii) are less obvious. Let us prove directly that the **first distributive law** in (v) is valid, that is, we will prove:

Proposition : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof : We will show that $x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)$. To this end, observe that:

$$\begin{aligned}
 x \in A \cap (B \cup C) &\iff (x \in A) \text{ AND } [x \in (B \cup C)] \\
 &\iff (x \in A) \text{ AND } [(x \in B) \text{ OR } (x \in C)] \\
 &\left\{ \begin{array}{l} \text{Corresponding to the second term} \\ [(x \in B) \text{ OR } (x \in C)], \text{ there are two cases:} \\ \\ \text{Case 1: } (x \in B). \text{ In this case we have} \\ (x \in A) \text{ AND } (x \in B), \\ \text{that is, } x \in (A \cap B) \\ \\ \text{OR} \\ \\ \text{Case 2: } (x \in C). \text{ In this case we have} \\ (x \in A) \text{ AND } (x \in C), \\ \text{that is, } x \in (A \cap C) \end{array} \right. \\
 &\iff [\text{Case 1}] \text{ OR } [\text{Case 2}] \\
 &\iff [x \in (A \cap B)] \text{ OR } [x \in (A \cap C)] \\
 &\iff x \in (A \cap B) \cup (A \cap C)
 \end{aligned}$$

Remark : Those of you who are really sharp (or pedantic) will have observed that the split into **Case 1** and **Case 2** above is a little confusing, because these cases are not, in fact, separate. Indeed, the condition that $[x \in (B \cup C)]$ splits into **three separate cases**, namely:

$$x \in (B \cup C) \iff [x \in (B \cap \bar{C})] \text{ OR } [x \in (\bar{B} \cap C)] \text{ OR } [x \in (B \cap C)]$$

That is we have written the set $(B \cup C)$ as the following **disjoint union** of sets:

$$(B \cup C) = (B \cap \bar{C}) \cup (\bar{B} \cap C) \cup (B \cap C),$$

It is this level of detail (or pedantry, as some might call it) that leads us to our next topic.

The Disjunctive Normal Form: As motivation, supposed that we wish to answer some question relating to two sets, A and B . We begin by splitting the universal set U into disjoint sets depending on whether or not each element

$$u \in U \text{ is in } (A \text{ or in } \bar{A}) \text{ and whether or not } u \text{ is in } (B \text{ or in } \bar{B}).$$

We note that for each $u \in U$, there are 2 choices as to whether u is in $(A \text{ or in } \bar{A})$ and again there are 2 choices as to whether u is in $(B \text{ or in } \bar{B})$. That is a total of $2 \times 2 = 4$ choices so that U is split into a **disjoint union** of **four** sets as follows:

$$U = (A \cap B) \cup (\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}).$$

Diagram 1.

We can think of these disjoint subsets of U as the basic building blocks of U when it comes to questions about A and B . In particular, any set which is determined by A and B can be written **uniquely** as a disjoint union of some of these subsets. For example,

$$A = (A \cap B) \cup (A \cap \bar{B}).$$

Diagram 2.

and

$$(A \cup B) = (A \cap B) \cup (\bar{A} \cap B) \cup (A \cap \bar{B}).$$

Diagram 3.

When sets are written in this way (relative to A and B) we say that they are written in their **disjunctive normal form** (relative to A and B).

If we were dealing with three sets A , B and C , then the universal set U would split into $2 \times 2 \times 2 = 8$ disjoint subsets. This is because each $u \in U$ is;

in (either A **or** \bar{A}), 2 choices here,
and
in (either B **or** \bar{B}), 2 choices here,
and
in (either C **or** \bar{C}), 2 choices here.

That is a total of $2 \times 2 \times 2 = 8$ choices.

If we are given two complicated expressions involving sets, then one sure way to determine whether or not these expressions correspond to the same set is to write out the sets corresponding to each of the expressions in their disjunctive normal forms. We illustrate this use of the disjunctive normal form by using it to prove the first of De Morgan's Laws.

Proof of the first De Morgan's Law: That is, a proof of the statement:

$$\overline{(A \cap B)} = \bar{A} \cup \bar{B}.$$

We begin, as explained above, by writing U as a **disjoint union**:

$$U = (A \cap B) \cup (\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B}).$$

From this, it is clear that

$$\overline{(A \cap B)} = (\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})$$

and, furthermore,

$$\begin{aligned}
\overline{A \cap B} &= [\overline{A}] \cup [\overline{B}] \\
&= [(\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})] \cup [(\overline{B} \cap A) \cup (\overline{B} \cap \overline{A})] \\
&= (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) \cup (\overline{B} \cap A) \\
&= (\overline{A} \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap \overline{B}).
\end{aligned}$$

That is, $\overline{(A \cap B)}$ and $\overline{A} \cup \overline{B}$ have the same disjunctive normal form and, therefore, must be equal.

Remark : We can use the identity $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ (in conjunction with the fact that $\overline{\overline{A}} = A$ for any set A) to show that

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}.$$

To see this, simply observe that

$$\begin{aligned}
\overline{(A \cup B)} &= \overline{(\overline{\overline{A} \cup \overline{B}})} \\
&= \overline{\overline{(\overline{A} \cap \overline{B})}}, \quad \text{because } \overline{A} \cup \overline{B} = \overline{(A \cap B)}, \\
&= (\overline{A} \cap \overline{B}).
\end{aligned}$$

In addition to using the disjunctive normal forms of sets we can also use laws (i) to (ix) above to simplify complicated expressions for sets.

Example :

$$\begin{aligned}
\overline{\overline{A} \cap \overline{[(B \cup C) \cap A]}} &= \overline{\overline{A}} \cup \overline{\overline{[(B \cup C) \cap A]}} \quad \text{by the first law in (vii)} \\
&= A \cup \overline{[(B \cup C) \cap A]} \quad \text{by law (vi)} \\
&= A \cup [A \cap (B \cup C)] \quad \text{by the first law in (ii)} \\
&= A \quad \text{by the second law in (iv)}
\end{aligned}$$